Practice questions

1. A power utility can supply electricity to a city from n different power plants. Power plant i fails with probability p_i, independent of the others.

   (a) Suppose that any one plant can produce enough electricity to supply the entire city. What is the probability that the city will experience a black-out?

   (b) Suppose that all n plants are necessary to produce enough electricity to supply the entire city. What is the probability that the city will experience a black-out?

   (Adapted from Textbook problem 1.36)

   Solution:

   (a) The city experiences a blackout if all plants fail, namely the blackout event B is the intersection of all plant failures $F_1, \ldots, F_n$. By independence, $\mathrm{P}(B) = \mathrm{P}(F_1 \cap \cdots \cap F_n) = \mathrm{P}(F_1) \cdots \mathrm{P}(F_n) = p_1 \cdots p_n$.

   (b) In this city, the blackout event B' is the union of all plant failures. A blackout does not happen if all plants are operational, namely $B'^c = F_1^c \cap \cdots \cap F_n^c$. By independence, $\mathrm{P}(B'^c) = \mathrm{P}(F_1^c) \cdots \mathrm{P}(F_n^c)$, and from the axioms of probability we get that $\mathrm{P}(B') = 1 - \mathrm{P}(B'^c) = 1 - (1 - p_1) \cdots (1 - p_n)$.

2. An ENGG 2430A tutorial meets for 11 weeks. Each week, the TA asks 5 questions and chooses 5 random but distinct students to answer them, independently of what happened in previous weeks. If you are one of 30 students in the tutorial (and attendance is always perfect!), what is the probability that you are chosen in the final week but not before that?

   Solution: The probability that you are chosen in any given week is 5/30, so the probability that you are not chosen in that week is 25/30. The week in which you are first chosen is a Geometric(5/30) random variable, so the probability it takes value 11 is $(25/30)^{10} \cdot 5/30 \approx 0.0269$.

3. You go to a party with 500 guests.

   (a) What is the probability that exactly one other guest has the same birthday as you? (For simplicity, exclude birthdays on February 29.)

   (b) Now model the number of other guests that share your birthday as a Poisson(λ) random variable N. What is the rate λ? How does the probability that N equals 1 compare to the answer in part (a)?

   (Adapted from Textbook problem 2.2)

   Solution:

   (a) We can model the number of guests having *your* birthday as a Binomial($n = 499, p = 1/365$) random variable X. The probability that $X = 1$ is $\binom{499}{1} \cdot p \cdot (1-p)^{499-1} \approx 0.3487$.

   (b) We can model this process as a Poisson(λ) random variable N with $\lambda = np = 499/365$. Then the probability of $N = 1$ is $\lambda \cdot e^{-\lambda} \approx 0.3484$, which is slightly smaller than the answer in part (a).

4. Alice, Bob, and Charlie are equally likely to have been born on any three days of the year. Let E_{AB} be the event that Alice and Bob were born on the same day. Define E_{BC} and E_{CA} analogously. Which of the following statements is true:

   (a) Any two of the three events E_{AB}, E_{BC}, E_{CA} are independent.

   (b) E_{AB}, E_{BC}, and E_{CA} are independent.

   (c) E_{AB} and E_{BC} are independent conditioned on E_{CA}.

   Solution: Our sample space will consist of all triples of possible birthdays (a, b, c) where a, b, and c are numbers between 1 and 365 (we exclude February 29 to keep things simple). We assume equally likely outcomes, so all triples occur with probability 365^{-3}.

   (a) **True.** The intersection of any two events is the event that all three were born on the same day. There are 365 such outcomes, each occurring with probability 365^{-3}, so

   $$P(E_{AB} \cap E_{BC}) = P(E_{AB} \cap E_{CA}) = P(E_{BC} \cap E_{CA}) = 365^{-2}.$$

   On the other hand, probability that any two of them were born on the same day is

   $$P(E_{AB}) = P(E_{BC}) = P(E_{CA}) = 365 \cdot \frac{1}{365^2} = 365^{-1}.$$

   Since $P(E_{AB} \cap E_{BC}) = 365^{-2} = P(E_{AB}) \cdot P(E_{BC})$, the two events E_{AB} and E_{BC} are independent, and similarly for the other two pairs.

   (b) **False.** $E_{AB} \cap E_{BC} \cap E_{CA}$ is also the event that all three were born on the same day, so $\mathrm{P}(E_{AB} \cap E_{BC} \cap E_{CA}) = 365^{-2}$. On the other hand $P(E_{AB}) \cdot P(E_{BC}) \cdot P(E_{CA}) = 365^{-3}$ so the three events are not independent.

   (c) **False.** Conditional independence holds when

   $$\mathrm{P}(E_{BC}|E_{CA}) = \mathrm{P}(E_{BC}|E_{CA} \cap E_{AB}).$$

   The probability on the left is the ratio of the probabilities of $E_{BC} \cap E_{CA}$ and E_{CA}, so it equals 365^{-1}. The probability on the right equals one, because if Alice and Bob share a birthday and Alice and Charlie also do, so will Bob and Charlie.

5. You flip two fair coins. If they both come out heads you stop. If not, you try again until they do. Let F be the total number of coin flips you performed. For example if the outcome is `TH HT HH` then $F = 6$. What is the PMF (probability mass function) of F?

   Solution: F can never be odd as you always perform an even number of flips. To perform a total of $2k$ flips (k rounds), the first $k-1$ rounds must have all resulted in failure and the last one in success. The probability of each round succeeding is $1/4$ and the successes are independent of one another, so

   $$\mathrm{P}(F = 2k) = (3/4)^{k-1}(1/4),$$

   where k ranges over $1, 2, \ldots$, or if you prefer

   $$\mathrm{P}(F = f) = (3/4)^{f/2-1}(1/4),$$

   where f ranges over the nonnegative even numbers.