

1. Alice tosses two fair coins. Given that at least one is a head, what is the probability that both are heads?

Solution: Let E and F be the events that at least one is a head and both are heads, respectively. We want to know the probability of $P(F|E)$. Then,

$$P(E) = 1 - P(E^c) = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

and

$$P(E \cap F) = P(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Therefore,

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{1}{3}.$$

2. Alice draws two cards at random from a deck of 52. Given that the first card is from a black suit, what is the probability that the second card is a spade (♠)?

Solution: Let B_1 and S_2 be the events that the first card is black and the second card is a spade, respectively. We want to calculate $P(S_2|B_1) = P(S_2 \cap B_1)/P(B_1)$. As half the cards are black, $P(B_1) = 1/2$. We can calculate $P(S_2 \cap B_1)$ using conditional probabilities. The probability $P(S_2)$ that the second card is a spade is $1/4$, and the probability $P(B_1|S_2)$ that the first card is black given that the second card is a spade is $25/51$ as after the “second” spade has been taken out there are 25 black cards left out of 51, all equally likely to be chosen as the “first” card. Therefore

$$P(S_2|B_1) = \frac{P(B_1|S_2)P(S_2)}{P(B_1)} = \frac{(25/51) \cdot (1/4)}{1/2} = \frac{25}{102}.$$

3. Alice tosses a six-sided die, then she tosses R fair coins, where R is roll of the die. Given that all the coin tosses came out tails, find the probabilities of each outcome for the die.

Solution: Let Y_i be the event that the roll of the die is i , and A be the event that all the coin tosses are tails. Then $P(A|Y_i) = (1/2)^i$. By Bayes’ rule,

$$P(Y_i|A) = \frac{P(A|Y_i) \cdot P(Y_i)}{P(A)} = \left(\frac{1}{2}\right)^i \cdot \frac{1/6}{P(A)}.$$

Since these probabilities must add up to one, it must be that

$$P(Y_i|A) = \frac{(1/2)^i}{(1/2)^1 + (1/2)^2 + \dots + (1/2)^6},$$

$$P(Y_1|A) = \frac{32}{63}, P(Y_2|A) = \frac{16}{63}, P(Y_3|A) = \frac{8}{63}, P(Y_4|A) = \frac{4}{63}, P(Y_5|A) = \frac{2}{63}, \text{ and } P(Y_6|A) = \frac{1}{63}.$$

4. There are 6 red balls and 1 blue ball. Each ball is randomly placed in one of two bins.
 - (a) What is the probability that the bin with the larger number of balls contains k balls ($k \in \{4, 5, 6, 7\}$)?
 - (b) What is the probability that the bin with the larger number of balls contains the blue ball? (*Hint:* Use Bayes’ rule.)

Solution: (a) The sample space Ω consists of all sequences of length 7, where the value of each position can be either 1 or 2, denoting which bin the ball goes to. Ω has size 2^7 . Let E_k denote the event the bin with the larger number of balls contains k balls. Then E_k consists of strings that contain k 1s and $7 - k$ 2s, or k 2s and $7 - k$ 1s. Therefore $|E_k| = \frac{\binom{7}{k} + \binom{7}{7-k}}{2^7} = \frac{\binom{7}{k}}{2^6}$. For $k \in \{4, 5, 6, 7\}$, we have

$$P(E_k) \begin{array}{c|cccc} & 4 & 5 & 6 & 7 \\ \hline & \frac{35}{64} & \frac{21}{64} & \frac{7}{64} & \frac{1}{64} \end{array}$$

(b) Let B denote the event the larger bin contains the blue ball. We have $P(B | E_k) = \frac{k}{7}$. Using the rule of average conditional probabilities, we have

$$P(B) = P(E_4)P(B | E_4) + P(E_5)P(B | E_5) + P(E_6)P(B | E_6) + P(E_7)P(B | E_7).$$

Plugging in the values we get $P(B) = \frac{21}{32}$.

5. Jar A contains 10 black balls and jar B contains 10 white balls. At each step, a ball is picked at random from each jar and moved to the other jar (so the number of balls in each jar stays the same). What is the probability that after four steps the initial configuration is recovered? (*Textbook problem 1.23*)

Solution: Let $p_t(i)$ be the probability that after t steps, jar A contains i white balls. We want to know $p_4(0)$. By the total probability theorem,

$$p_4(0) = \left(\frac{1}{10}\right)^2 \cdot p_3(1)$$

because conditioned on jar A having exactly one white after 3 steps, the initial configuration is recovered with probability $\left(\frac{1}{10}\right)^2$, and otherwise the initial configuration cannot be recovered. By the same reasoning,

$$p_3(1) = \left(\frac{2}{10}\right)^2 \cdot p_2(2) + 2 \cdot \frac{9}{10} \cdot \frac{1}{10} \cdot p_2(1) + 1 \cdot p_2(0).$$

After one step, jar A must have exactly one white ball, so

$$p_2(0) = \left(\frac{1}{10}\right)^2, \quad p_2(1) = 2 \cdot \frac{9}{10} \cdot \frac{1}{10}, \quad p_2(2) = \left(\frac{9}{10}\right)^2$$

Plugging in we get that $p_3(1) = 0.0748$ and so $p_4(0) = 0.000748$.