Practice Midterm 1

1. 3 red balls and 3 blue balls are randomly arranged on a line. Let X be the position of the first blue ball. (E.g. for the arrangement `RBRBBR`, $X = 2$.) Find the probability mass function of X.

   Solution: The sample space consists of all arrangements of 3 red balls and 3 blue balls. We assume equally likely outcomes. The random variable X takes integer values from 1 to 4. X takes value x when the first $x-1$ balls are red and the x-th ball is blue; the remaining $6-x$ balls must then contain exactly two blue balls. Using the equally likely formula, we get that

$$P(X = x) = \frac{\binom{6-x}{2}}{\binom{6}{3}} = \frac{(6-x)(5-x)}{40}$$

   or, in tabular form,

x	1	2	3	4
$P(X = x)$	1/2	3/10	3/20	1/20

2. Half the students know the answer to a true-false question. The other half guesses at random. I ask a random student and his answer is correct. What is the probability he knows the answer?

   Solution: The sample space consists of all students under equally likely outcomes. Let K be the event a student knows the answer, C be the event his answer is correct. We have $\mathrm{P}(K) = 1/2, \mathrm{P}(C \mid K) = 1, \mathrm{P}(C \mid K^{\mathsf{c}}) = 1/2$. By Bayes' rule

$$\mathrm{P}(K \mid C) = \frac{\mathrm{P}(C \mid K)\,\mathrm{P}(K)}{\mathrm{P}(C \mid K)\,\mathrm{P}(K) + \mathrm{P}(C \mid K^{\mathsf{c}})\,\mathrm{P}(K^{\mathsf{c}})}.$$

   Plugging in the values we get $\mathrm{P}(K \mid C) = 2/3$.

3. Toss a coin 4 times. Let X, Y and Z be the number of heads among the first two, middle two, and last two tosses, respectively. Are X and Z independent given that $Y \neq 1$? Justify carefully.

   Solution: No. Given that $Y \neq 1$, the events $Y = 0$ and $Y = 2$ (two tails and two heads in the middle each happen with probability) half. By the total probability theorem,

$$\mathrm{P}(X = 0 | Y \neq 1) = \frac{1}{2}\mathrm{P}(X = 0 | Y = 0) + \frac{1}{2}\mathrm{P}(X = 0 | Y = 2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{4}.$$

   By symmetry, $\mathrm{P}(Z = 0 | Y \neq 1)$ is also $1/4$. However,

$$\begin{aligned}\mathrm{P}(X = 0, Z = 0 | Y \neq 1) &= \frac{1}{2}\mathrm{P}(X = 0, Z = 0 | Y = 0) + \frac{1}{2}\mathrm{P}(X = 0, Z = 0 | Y = 2) \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 0 \\ &= \frac{1}{8},\end{aligned}$$

so the events $X = 0$ and $Z = 0$ are not independent conditioned on $Y = 1$. Therefore X and Z are not conditionally independent.

4. The average lifetime of a lightbulb is 10 months. You install 10 lightbulbs today. What is the probability that at least one of them fails within a month? Assume their failures are independent.

   Solution: To come up with a model for lightbulb failure, we partition each month into intervals of length $1/n$ (in months) for some large n. For every such interval, given that the lightbulb has not failed in the previous intervals, a failure happens with probability p. Since on average, we expect one failure every 10 months, $10np = 1$, so $p = 1/10n$. As n becomes large, we can describe the number of failures of a single lightbulb within a month as a Poisson(1/10) random variable.

   The probability that none of the lightbulbs fail within a month is then the probability that 10 independent Poisson(1/10) random variables are all zero, which equals $((1/10)^0 e^{-1/10}/0!)^{10} = 1/e$. So the probability at least one fails is $1 - 1/e \approx 0.632$.

5. Eight people's hats are mixed up and randomly redistributed. What is the expected number of pairs that exchanged hats (Alice got Bob's and Bob got Alice's)?

   Solution: For every two people i and j we introduce a random variable X_{ij} that takes value 1 if the two exchanged hats and 0 if not. The expected value of X_{ij} is the probability that i and j exchanged hats, which is $1/8 \cdot 7$: Bob gets Alice's hat with probability 1/8, and given this happened Alice gets Bob's with probability 1/7. The number of pairs that exchanged hats is $X_{12} + X_{13} + \cdots + X_{78}$, where the indices range over all (ordered) $\binom{8}{2}$ pairs of people. By linearity of expectation,

$$\mathrm{E}[X_{12} + X_{13} + \cdots + X_{78}] = \mathrm{E}[X_{12}] + \mathrm{E}[X_{13}] + \cdots + \mathrm{E}[X_{78}] = \binom{8}{2} \cdot \frac{1}{8 \cdot 7} = \frac{1}{2}.$$

Practice Midterm 2

1. Let N be the number of times you flip a fair coin until you observe both a head and a tail. For example if the outcome is `HHHT` then $N = 4$. What is $\mathrm{E}[N]$ and $\mathrm{Var}[N]$?

   Solution: $N = 1 + G$ where G is a Geometric(1/2) random variable: If the first toss is head you keep tossing a fair coin until you see a tail, and vice versa. Therefore $\mathrm{E}[N] = 1 + \mathrm{E}[G] = 3$ and $\mathrm{Var}[N] = \mathrm{Var}[G] = (1 - 1/2)/(1/2)^2 = 2$.

2. You flip a p-biased coin (heads with probability p) 10 times. For which value(s) of p, if any, are the events $A =$ "the first two flips are heads" and $B =$ "there are exactly two heads" independent?

   Solution: . Event A has probability p^2. Event B has probability $\binom{10}{2} p^2(1-p)^8$. Event $A \cap B$ occurs when two heads are followed by eight tails, so it has probability $p^2(1-p)^8$. So A and B are independent if and only if

$$p^2 \cdot \binom{10}{2} p^2(1-p)^8 = p^2(1-p)^8.$$

   This equation has four solutions: $p = 0$, $p = 1$ and $p = \pm\binom{10}{2}^{-1/2} = \pm 1/\sqrt{45}$. Discarding the negative one, the desired values of p are 0, 1, and $1/\sqrt{45}$.

3. Keep rolling a 3-sided die until the sum of the values strictly exceeds 2. For example if the first roll is a 1 then you roll again; if the second roll is then a 2, $1 + 2 = 3 > 2$ and you stop. Find the PMF of the number of times you rolled.

   Solution: The random variable X of interest can take values 1, 2, and 3. $X = 1$ happens if and only if the first roll is a 3, so $\mathrm{P}(X = 1) = 1/3$. $X = 3$ happens if and only if the first two rolls are both 1, so $\mathrm{P}(X = 3) = 1/9$. Since probabilities must add up to 1, we get that $\mathrm{P}(X = 2) = 1 - 1/3 - 1/9 = 5/9$.

4. On a light rain day, rain falls at an average rate of 1 drop per second. On a heavy rain day, the average rate is 2 drops per second. 2/3 of the rainy days are light and 1/3 are heavy. You walk out and 2 drops of rain hit you in the next second. What is the probability it is a light rain day?

   Solution: Let L and H be the events of light and heavy rain and N be the number of drops that hit you. We model $N \mid L$ and $N \mid H$ as Poisson(1) and Poisson(2) random variables, respectively. By Bayes' rule,

$$\begin{aligned} \mathrm{P}(L|N = 2) &= \frac{\mathrm{P}(N = 2|L)\,\mathrm{P}(L)}{\mathrm{P}(N = 2|L)\,\mathrm{P}(L) + \mathrm{P}(N = 2|H)\,\mathrm{P}(H)} \\ &= \frac{(1^2 e^{-1}/2) \cdot (2/3)}{(1^2 e^{-1}/2) \cdot (2/3) + (2^2 e^{-2}/2) \cdot (1/3)} \\ &= \frac{e}{e + 2} \approx 0.576. \end{aligned}$$

5. Let N be the number of distinct values observed when a 6-sided die is rolled 6 times. For example, if the outcome is `521154` then the observed values are $\{1, 2, 4, 5\}$ and $N = 4$. What is $\mathrm{E}[N]$?

   Solution: We can write $N = N_1 + N_2 + N_3 + N_4 + N_5 + N_6$ where N_1 takes value 1 if a `1` was observed and 0 if not, and similarly for the others. Then $\mathrm{E}[N_1] = \mathrm{P}[N_1 = 1]$ is the probability that a `1` was observed. This is one minus the probability that a `1` was not observed, which is $(5/6)^6$ by independence. Summarizing, $\mathrm{E}[N_1] = 1 - (5/6)^6$. By symmetry, $\mathrm{E}[N_1] = \cdots = \mathrm{E}[N_6]$, so $\mathrm{E}[N] = 6(1 - (5/6)^6) \approx 3.991$.