This homework is optional. If you would like to have it graded please turn it your solution over email by Tuesday December 3. You are encouraged to collaborate on the homework and ask for assistance, but you are required to write your own solutions, list your collaborators, acknowledge any sources of help, and provide external references if you have used any.

Question 1

In Lecture 3 we showed that the inner product function $IP(x, y) = x_1y_1 + \cdots + x_ny_n \mod 2$, where $x, y \in \{0,1\}^n$ takes the same value on more than 7/8 of the entries of any set of the form $X \times Y$ where $|X| \cdot |Y| \geq K \cdot 2^n$ for some constant K. In this question you will show that the same is true with high probability for a random function $R\colon \{1, \dots, N\} \times \{1, \dots, N\} \to \{0, 1\}$, where $N = 2^n$.

(a) Let $Z_1, \dots, Z_M$ be a sequence of independent uniformly random coin tosses. Apply the inequality $\binom{M}{\delta M} \leq 2^{H(\delta)\cdot M}$ and a union bound to show that the probability more than $7M/8$ of the coins are heads is at most $2^{-M/4}$.

(b) Use part (a) to show that the probability R takes the same value on more than 7/8 of the entries of some set of the form $X \times Y$ is at most $2^{-|X|\cdot|Y|/4+1}$.

(c) Use part (b) and a union bound to show that a random function takes the same value on more than 7/8 of the entries of some set $X \times Y$ with $|X| \cdot |Y| \geq 9N$ with probability at most $2^{-\Omega(N)}$.

Question 2

Given an undirected graph G, let G^2 be the graph whose vertices are ordered pairs of vertices in G and whose edges are those pairs $\{(u, v), (u', v')\}$ such that $\{u, u'\}$ is an edge in G or $u = u'$, and $\{v, v'\}$ is an edge in G or $v = v'$.

(a) Show that if G has a clique of size k then G^2 has a clique of size k^2.

(b) Show that if G^2 has a clique of size K then G has a clique of size $\lceil \sqrt{K} \rceil$.

(c) Use parts (a) and (b) to show that if there exists a polynomial-time algortihm that finds a clique of size at 1% of the size of the largest clique in a graph, then there is a polynomial-time algorithm that finds a clique of size at least 99% the size of the largest clique.

Question 3

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *affine* if it is of the form $f(x) = \langle a, x\rangle + b$ for some $a \in \{0,1\}^n$ and $b \in \{0,1\}$. It is δ-far from affine if every affine function differs from it on more than a δ-fraction of inputs. The YES and NO instances of $(1, 1-\delta)$-GAP-AFFINE are functions that are affine and δ-far from affine, respectively.

(a) Let $g(x, y) = f(x) + f(y)$. Show that if f is affine then g is linear.

(b) Show that if g is δ-close to linear then f is δ-close to affine. (**Hint:** Fix y.)

(c) Use part (a) and results from Lecture 11 to show that the one-sided randomized query complexity of $(1, 1-\delta)$-GAP-AFFINE with error $1-\delta$ is at most 6.

(d) Show that for every three distinct points $x, y, z \in \{0,1\}^n$ and values $a, b, c \in \{0,1\}$ there exists an affine function f such that $f(x) = a$, $f(y) = b$, and $f(z) = c$.

(e) Use part (b) to show that the one-sided randomized query complexity of $(1, 1 - \delta)$-GAP-AFFINE with any error less than one is at least 4.

(f) (**Extra credit**) Show that "at most 6" can be replaced by "at most 4" in part (b).