ENGG 2440A: Discrete Mathematics for Engineers | Homework 6
The Chinese University of Hong Kong, Fall 2014 | due Friday 5 December

This homework is optional. If you turn it in for grading, you can use it to replace your lowest grade from the other five homeworks (provided that you followed the homework guidelines throughout the semester).

Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. In how many ways can 4 boys and 4 girls sit in a row if
   (a) there are no restrictions on the seating arrangement?
   (b) the boys and the girls are each to sit together?
   (c) only the boys must sit together?
   (d) no two people of the same sex are allowed to sit together?

2. This question concerns sequences of length k whose entries are numbers in the set $\{0, \ldots, n-1\}$.
   (a) How many such sequences are there?
   (b) How many such sequences are there in which at least two entries are the same?
   (c) If $n = 2$, how many sequences are there in which both a 0 and a 1 occur at least once?
   (d) **(Extra credit)** If $k = 100$ and $n = 20$, how many sequences are there in which every number from 0 to 19 occurs at least once?

3. Use the pigeonhole principle to prove the following propositions.
   (a) Among the 7 million people in Hong Kong, there must be three that live in the same district with the same number of hairs on their head. (Hong Kong has 18 districts. Assume a person can have at most 180,000 hairs.)
   (b) Among every 10 points inside an equilateral triangle of side length 1, there is a pair within distance 1/3.
   (c) Among every 15 integers between 0 and 100, there are two distinct pairs $\{x, y, z\}$, $\{x', y', z'\}$ of three numbers such that $x + y + z = x' + y' + z'$. (The three numbers in each pair are distinct, but the same number may belong to both pairs.)

4. You have 16 chopsticks in 8 different colours, two identical ones of each colour. In how many ways can you assign a pair of chopsticks to each of 8 guests so that no guest gets a matching pair? (**Hint:** Use the inclusion-exclusion formula and possibly a computer.)

5. Let X be the set of bit sequences of length $n-k$ with exactly k zeros and Y be the set of n bit sequences that have k zeros no two of which are in consecutive positions and that always end in a 1. For example, if $n=7$ and $k=3$ then $(0,1,1,0,1,0,1) \in Y$ but $(0,1,1,1,0,0,1), (0,1,0,1,0,1,0) \notin Y$.

   Let $f\colon X \to Y$ be the function that does the following operation to its input bits from left to right: (1) If the input bit is a 1, output a 1. (2) If the input bit is a 0 output the pair 0, 1. For example, $f(0,1,0,0) = (0,1,1,0,1,0,1)$.

   (a) Show that f is injective.

   (b) Show that f is surjective.

   (c) How many bit sequences are there with z zeros, o ones in which there is no pair of consecutive zeros?

   (d) (**Extra credit**) How many circular arrangements of z zeros and o ones are there in which there is no pair of consecutive zeros?

6. An *increasing subsequence* of a sequence $(s_1, \ldots, s_n)$ is a sequence of the type $(s_{i_1}, \ldots, s_{i_t})$ where $i_j < i_{j+1}$ and $s_{i_j} < s_{i_{j+1}}$ for all j from 1 to $t-1$. A *decreasing subsequence* is one in which $i_j < i_{j+1}$ and $s_{i_j} > s_{i_{j+1}}$ for all j from 1 to $t-1$. For example, if $s = (1,2,6,3,5)$ then $(2,3,5)$ is an increasing subsequence of s and $(6,5)$ is a decreasing subsequence of s.

   You are given a sequence of 101 distinct numbers $s = (s_1, \ldots, s_{101})$.

   (a) For every i between 1 and 101, let inc_i and dec_i be the longest increasing and decreasing subsequences of s that start with s_i respectively. (In the above example, $inc_2 = (2,3,5)$ and $dec_2 = (2)$. If there are several choices, take an arbitrary one.) Show that for every pair of entries $s_i \neq s_j$, either inc_i and inc_j are of different length or dec_i and dec_j are of different length.

   (b) Use the pigeonhole principle to show that s contains an increasing subsequence of length 11 or a decreasing subsequence of length 11.

   (c) Give an example of a sequence of length 100 that does not contain an increasing subsequence of length 11 or a decreasing subsequence of length 11.