Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. Use induction to prove the following statements.

   (a) For every $n \geq 4$, $n^2 \leq 2^n$.

   (b) For every $n \geq 1$, $1^2 - 2^2 + 3^2 - 4^2 + \ldots (+ \text{ or } -)\ n^2 = (-1)^{n+1} n(n+1)/2$.

   (c) For $n \geq 3$, the sum of interior angles of a convex polygon with n sides is $(n-2)\pi$. (You can assume that the base case $n = 3$ is true.)

2. Use strong induction to prove the following statements.

   (a) Every nonnegative integer has a binary representation: Every $n \geq 0$ can be written as $n = n_0 + 2n_1 + 4n_2 + \cdots + 2^k n_k$ for some $k \geq 0$, where $n_0, \ldots, n_k$ are 0 or 1.

   (b) $F_n \geq (3/2)^{n-2}$ for every $n \geq 1$, where the sequence F_n is defined by $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

3. You start with the three numbers -2, 0, and 2 and play the following game: In each round, you can take any pair of numbers a and b and replace them by $(a+b)/\sqrt{2}$ and $(a-b)/\sqrt{2}$. The third number stays the same. Can you ever end up with the numbers

   (a) 0, 0, and $\sqrt{8}$?

   (b) -2, 1, and 2?

   (c) $-\sqrt{3}$, $\sqrt{2}$, and $\sqrt{3}$?

4. Prove the following statements.

   (a) For all $n \geq 5$, n is composite if and only if n divides $(n-1)!$.

   (b) For all n, if $2^n - 1$ is a prime, then n is a prime.

   (c) (**Extra Credit**) There exists a positive integer n such that $n/2$ is a square, $n/3$ is a cube, and $n/5$ is a fifth power of an integer. (Do not just write down a number; explain how you discovered it.)

5. Show that the Extended Euclid's Algorithm $X(n, d)$ (see the Lecture 4 notes) terminates and outputs integers (s, t) such that $s \cdot n + t \cdot d = \gcd(n, d)$ whenever $n > d \geq 0$.

6. You will show that

$$S(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is *not* an integer for every $n \geq 2$.

(a) Express the sum $S(5)$ as a fraction. Show that the denominator does not divide the numerator. (It may be easier if you do *not* simplify the numerator.)

(b) Let $n \geq 2$ and 2^m be the largest power of 2 between 1 and n. (For example, if $n = 6$ then $m = 2$ and $2^m = 4$.) Show that 2^m does not divide any number between 1 and n except itself.

(c) Prove that $S(n)$ is not an integer for every $n \geq 2$.