ENGG 2430 / ESTR 2004: Probability and Statistics

Spring 2019

7. Continuous Random Variables II

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One random variable review

	PMF $f(x)$	PDF $f(x)$
$P(X \leq a)$	$\sum_{x \leq a} f(x)$	$\int_{x \leq a} f(x) dx$
$E[X]$	$\sum_{x} x f(x)$	$\int_{x} x f(x) dx$
$E[X^2]$	$\sum_{x} x^2 f(x)$	$\int_{x} x^2 f(x) dx$
$Var[X]$	$E[(X - E[X])^2] = E[X^2] - E[X]^2$	

Discrete two variables review

joint PMF $f_{XY}(x, y) = P(X = x, Y = y)$

Probability of A	$P(A) = \sum_{(x,y)\in A} f_{XY}(x,y)$
Derived RV $Z = g(X, Y)$	$f_Z(z) = \sum_{x,y:\, g(x,y)=z} f_{XY}(x,y)$
Marginals	$f_X(x) = \sum_y f_{XY}(x,y)$
Independence	$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.
Expectation	$E[Z] = \sum g(x,y) f_{XY}(x,y)$

Continuous random variables

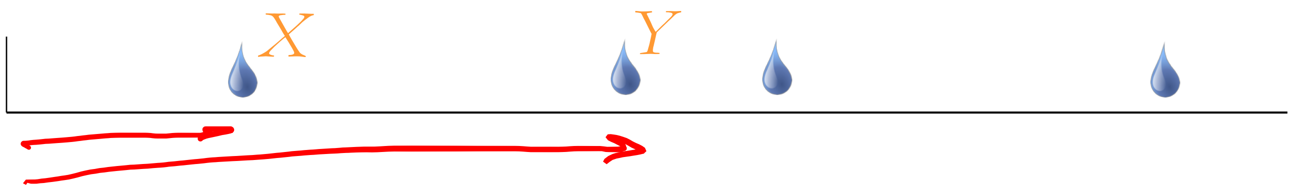
A pair of continuous random variables X, Y can be specified either by their joint c.d.f.

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

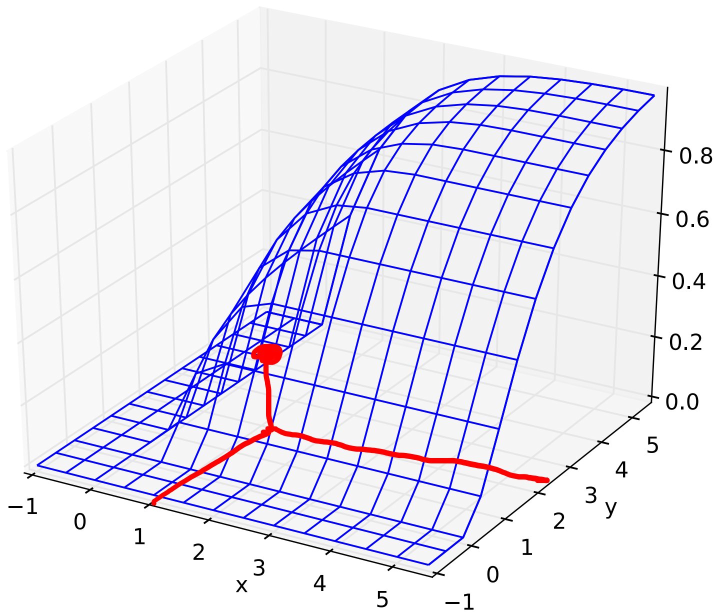
or by their joint p.d.f.

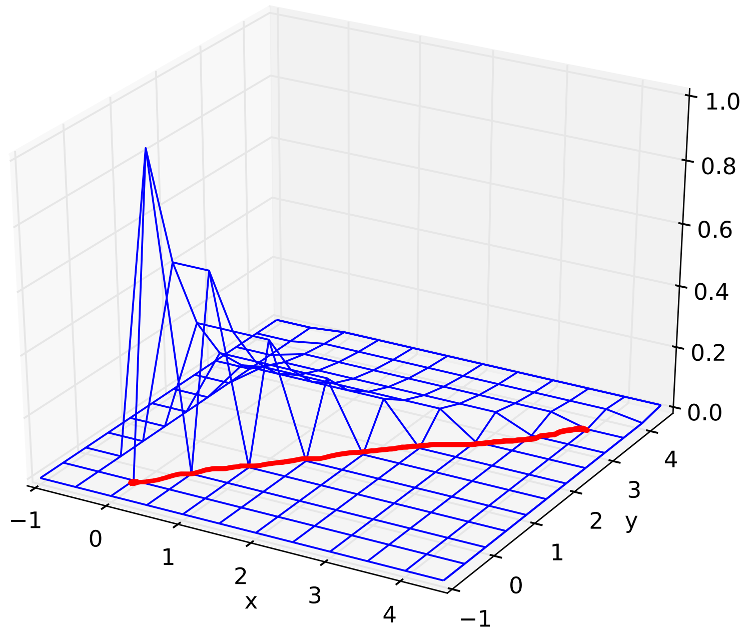
$$\begin{aligned} f_{XY}(x, y) &= \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x, y) \\ &= \lim_{\varepsilon, \delta \to 0} \frac{P(x < X \leq x + \varepsilon, y < Y \leq y + \delta)}{\varepsilon\delta} \end{aligned}$$

Rain drops at a rate of 1 drop/sec. Let X and Y be the arrival times of the first and second raindrop.



$Y \geq X$





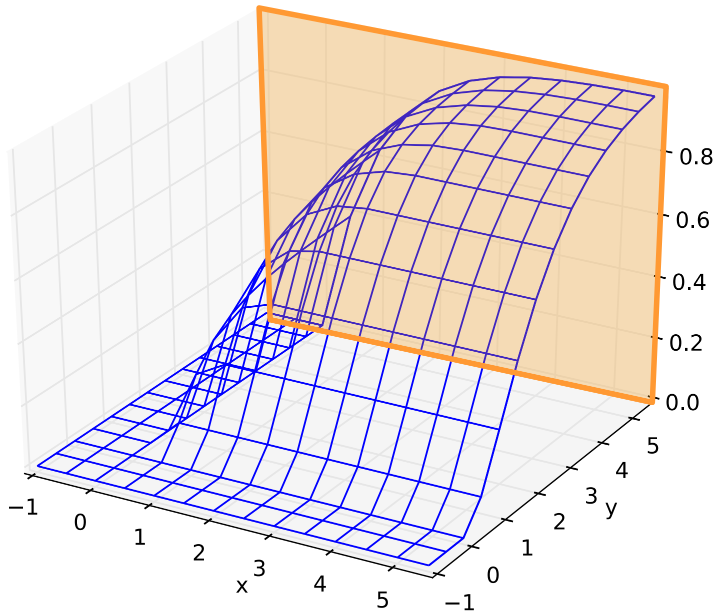
$$F(x,y) = P(X \leq x,\ Y \leq y)$$

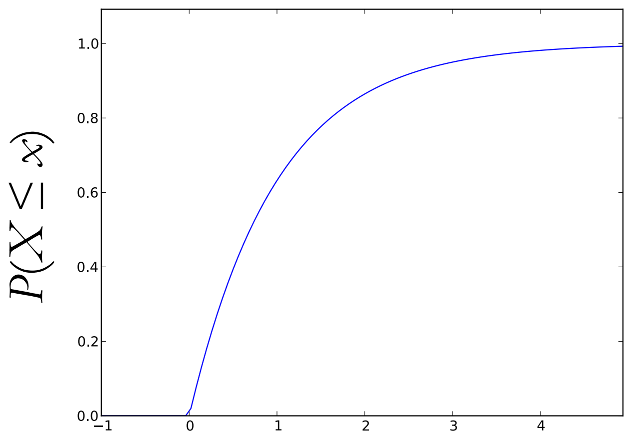
$$f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x,y)$$

Continuous marginals

Joint CDF $F_{XY}(x, y) = P(X \leq x, Y \leq y)$

Marginal CDF: $F_X(x) = P(X \leq x)$





Exponential(1)

the continuous cheat sheet

X, Y **continuous with joint p.d.f.** $f_{XY}(x, y)$

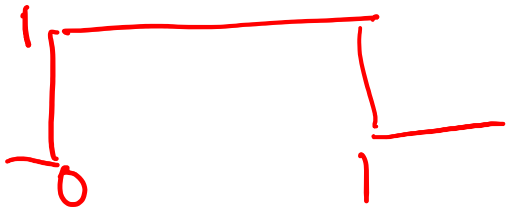
Probability of A	$P(A) = \iint_A f_{XY}(x, y)\ dxdy$
Derived RV $Z = g(X, Y)$	$f_Z(z) = \int_{(x, y):\ g(x, y) = z} f_{XY}(x, y)\ dxdy$
Marginals	$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\ dy$
Independence	$f_{XY}(x, y) = f_X(x)\ f_Y(y)$ **for all** x, y
Expectation	$E[Z] = \iint g(x, y)\ f_{XY}(x, y)\ dxdy$

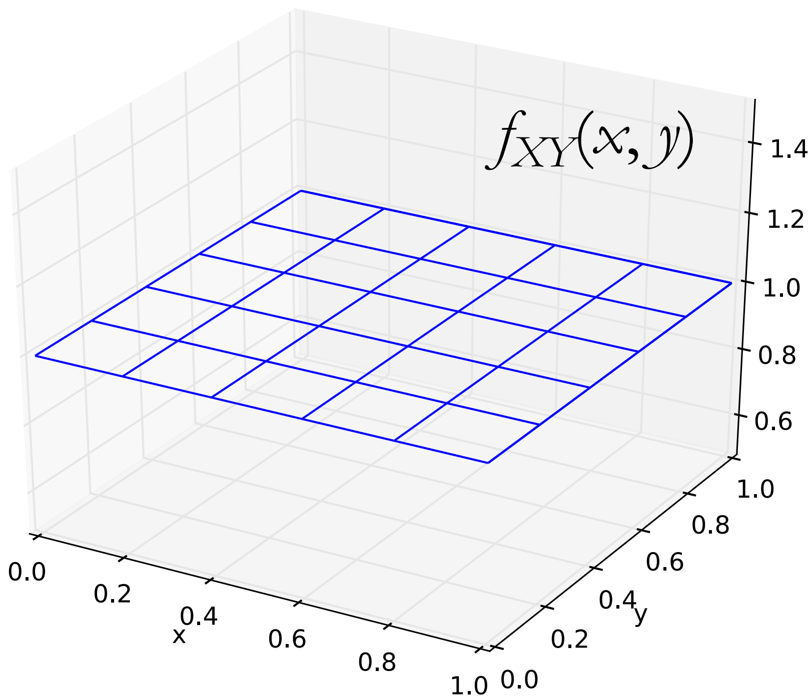
Independent uniform random variables

Let X, Y **be independent** $\text{Uniform}(0, 1)$**.**

$$f_{XY}(x, y) = f_X(x)\, f_Y(y) = \begin{cases} 1 & \textbf{if } 0 < x, y < 1 \\ 0 & \textbf{if not} \end{cases}$$

$f_X(x)$



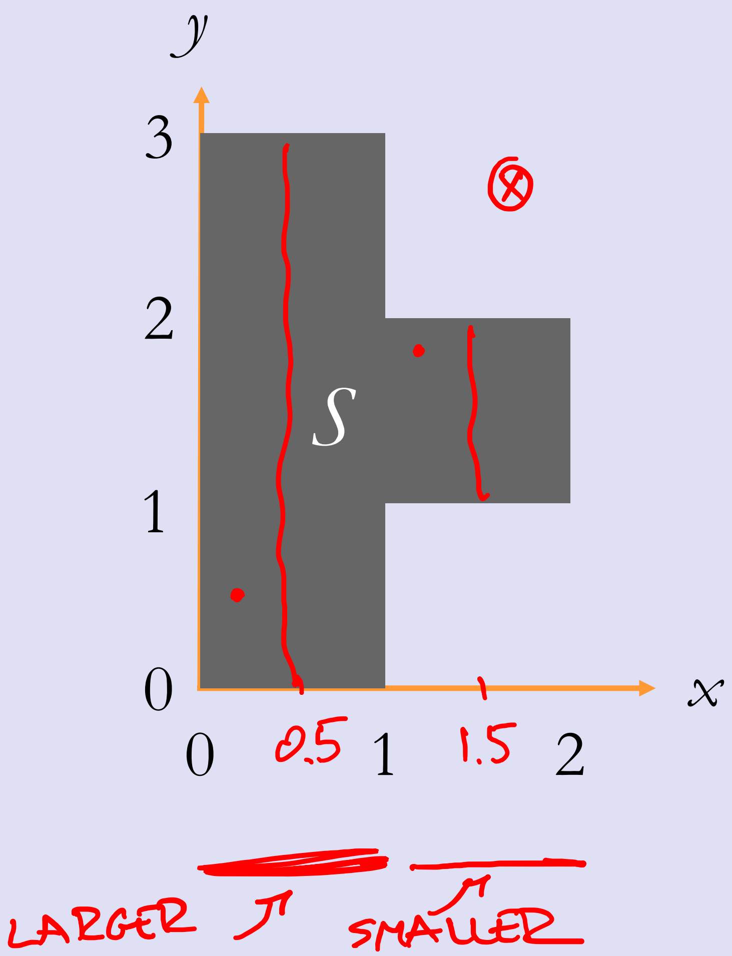


$$\mathbf{P}(A) = \iint_{(x,y)\in A} f(x,y)\,dx\,dy$$

$$= \text{area}(A)$$

Joint PDF of X, Y is uniform over S.

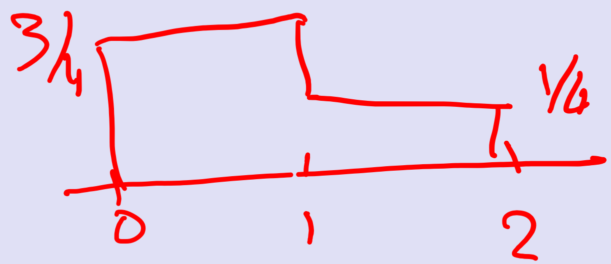
What are the marginals?



$$f_{XY}(x,y) = \begin{cases} 1/4 & \text{IF } (x,y) \in S \\ 0 & \text{IF NOT} \end{cases}$$

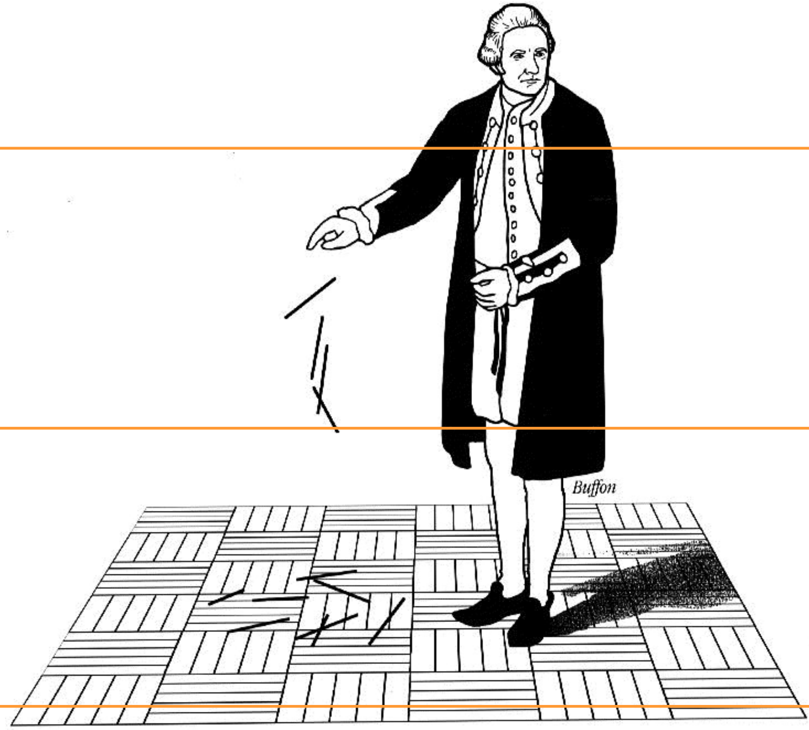
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \begin{cases} 3/4 & \text{IF } x \in (0,1) \\ 1/4 & \text{IF } x \in (1,2). \end{cases}$$



Buffon's needle

A needle of length l is randomly dropped on a ruled sheet.



What is the probability that the needle hits one of the lines?

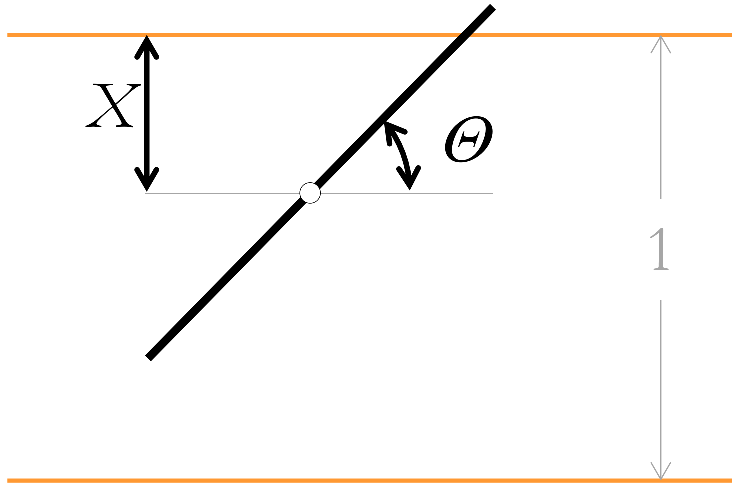
Probability model

X = dist to closest line

0 TO 1/2

Θ = angle

0 TO π



X Uniform(0, 1/2)

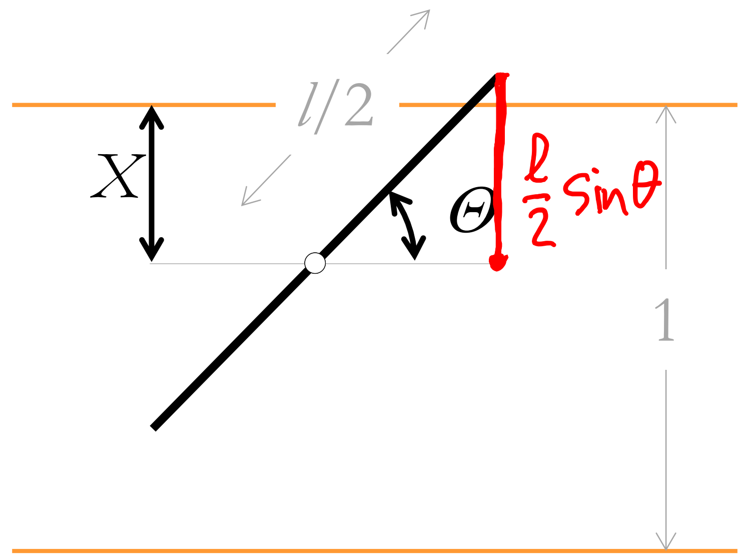
Θ Uniform(0, π)

X, Θ INDEPENDENT

Buffon's needle

PDF:

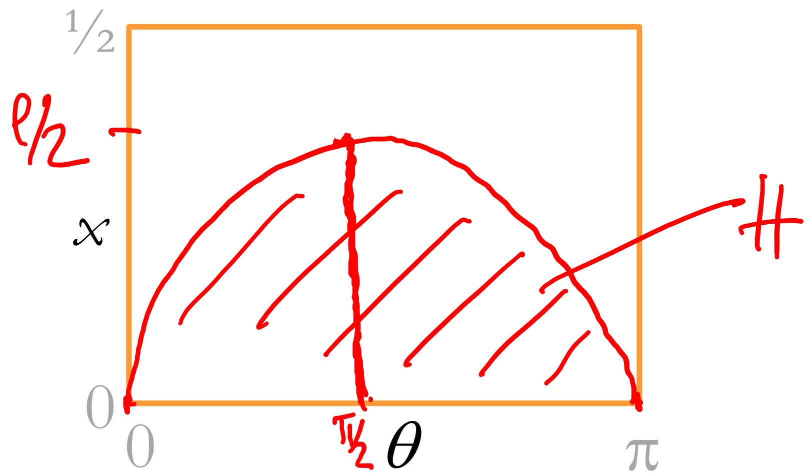
$$f_{X,\Theta}(x,\theta) = \frac{2}{\pi}$$



Event H **:** $\ell \le 1$

$$\frac{\ell}{2}\sin\theta \ge x$$

$$H = \left\{(\theta, x) : x \le \frac{\ell}{2}\sin\theta\right\}$$



Buffon's needle

Assume $l \leq 1$ (short needle)

$$\mathbf{P}(H) = \iint_{(x,\theta)\in H} f_{X\Theta}(x,\theta)\, dx\, d\theta$$

$$= \iint_{(x,\theta)\in H} \frac{2}{\pi}\, dx\, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{x=0}^{\ell/2 \sin\theta} \frac{2}{\pi}\, dx\, d\theta$$

$$= \int_{\theta=0}^{\pi} \frac{2}{\pi} \cdot \frac{\ell}{2} \sin\theta\, d\theta$$

$$= \frac{\ell}{\pi} \underbrace{\int_{\theta=0}^{\pi} \sin\theta\, d\theta}_{2} = \frac{2\ell}{\pi}$$

Conditioning

X \ Y	discrete	continuous
discrete	$\mathbf{P}(X = x \mid Y = y) = \dfrac{\mathbf{P}(X = x, Y = y)}{\mathbf{P}(Y = y)}$	
continuous		

Conditioning

X \ Y	discrete	continuous
discrete		
continuous		

$$\mathbf{P}(X \leq x \mid A) = \frac{P(X \leq x \text{ AND } A)}{P(A)}$$

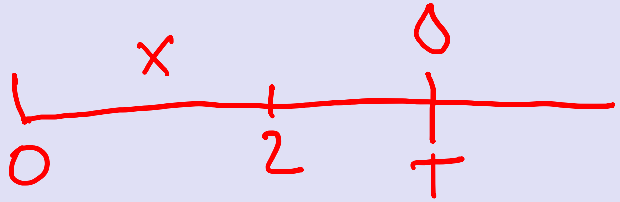
$$\mathbf{P}(X \leq x \mid Y \leq y) = \frac{P(X \leq x \text{ AND } Y \leq y)}{P(Y \leq y)}$$

Rain drops at a rate $\lambda = 1/\text{sec}$.

You walk for 2 sec, no drop yet.

What is the arrival time of next drop?

FIRST

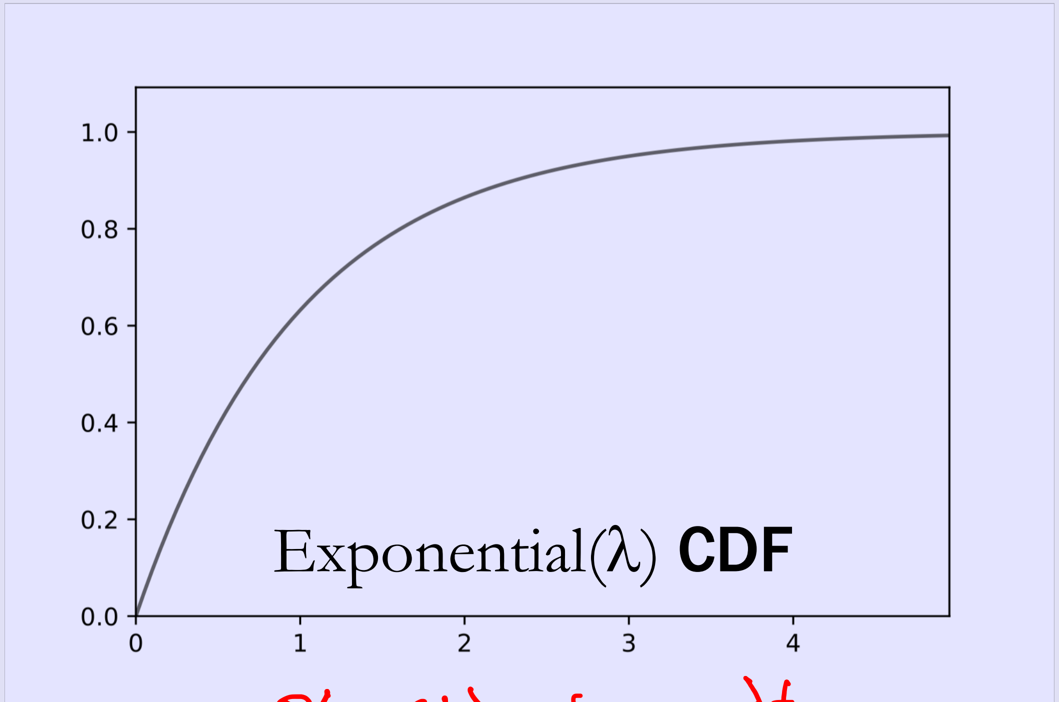


$$P(T>t \mid T>2)$$

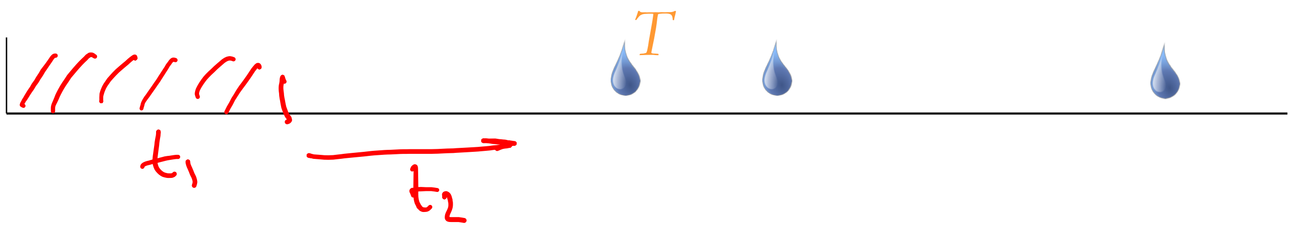
$$= \frac{P(T>t \text{ AND } T>2)}{P(T>2)}$$

$$= \frac{P(T>t)}{P(T>2)} = \frac{e^{-\lambda t}}{e^{-\lambda 2}}$$

$$= e^{-\lambda(t-2)}$$

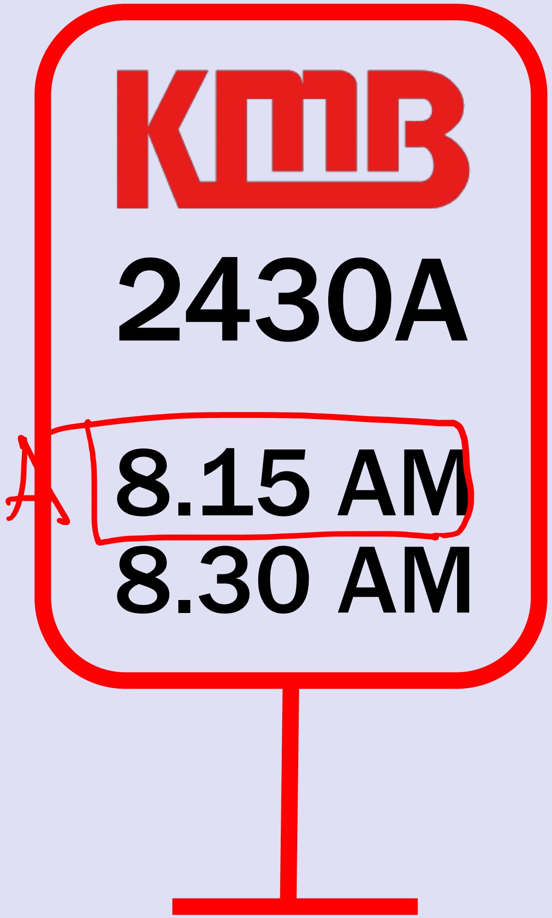


$$P(T \le t) = 1 - e^{-\lambda t}$$



Memorylessness of Exponential(λ) **RV:**

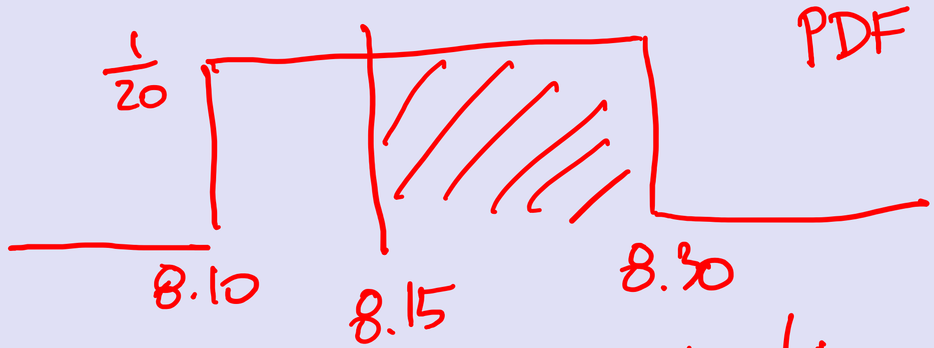
$$P(T > t_1 + t_2 \mid T > t_1) = P(T > t_2)$$



Alice arrives 8.10 - 8.30.

Given she caught the first bus, what is her arrival time?

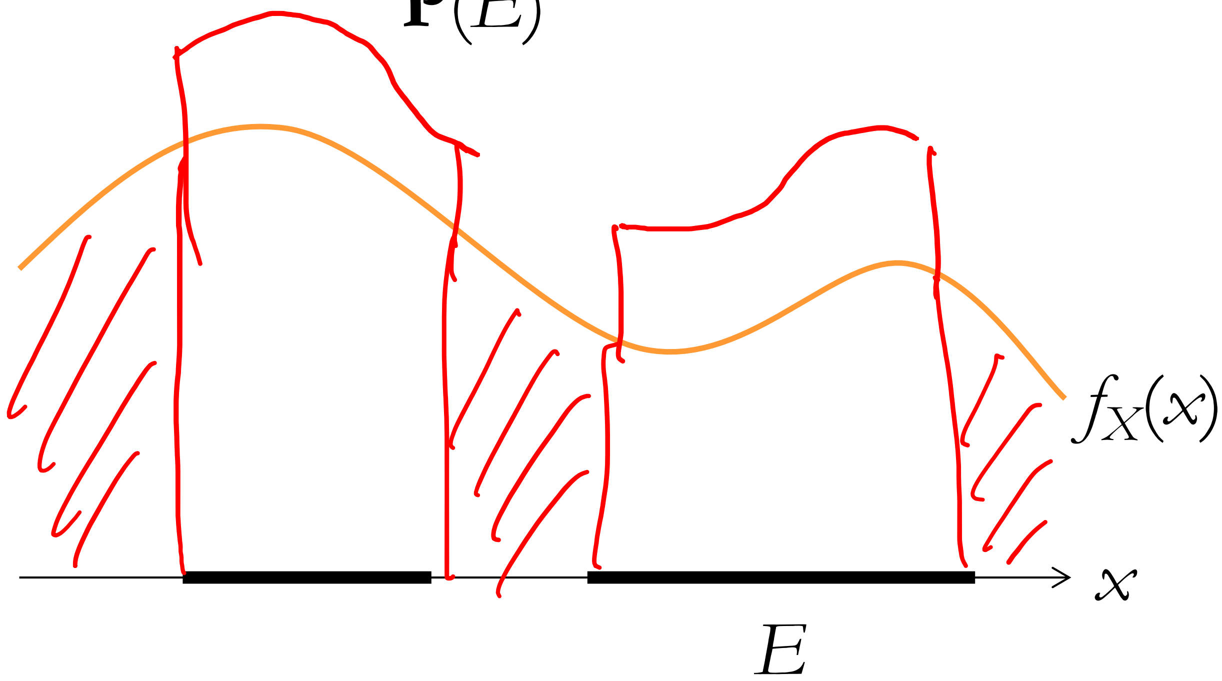
MODEL: T Uniform(8.10, 8.30)



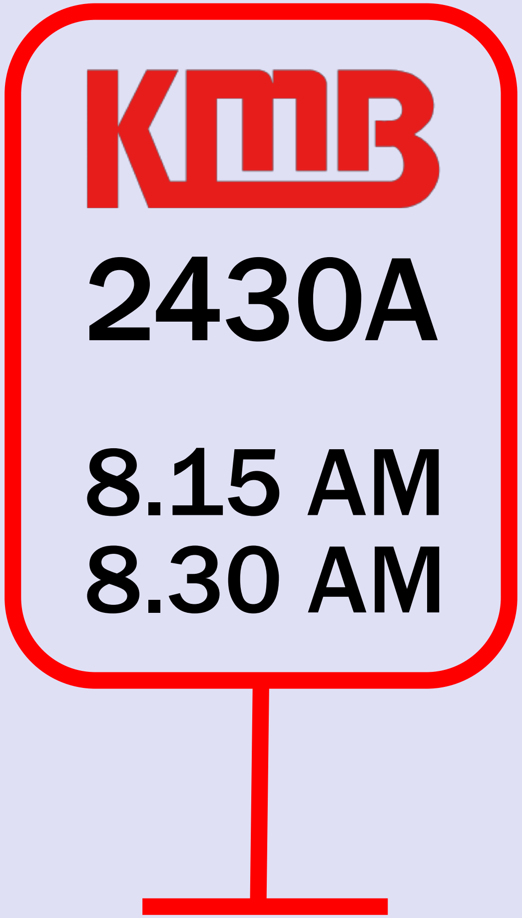
CONDITIONAL PDF T|A

Conditioning a continuous RV on an event

PDF: $f_{X \mid E}(x) = \dfrac{f_X(x)}{\mathbf{P}(E)}$ **when** $x \in E$



CDF: $\mathbf{P}(X \leq x) = \int_{-\infty}^{x} f_{X|E}(t)\,dt.$

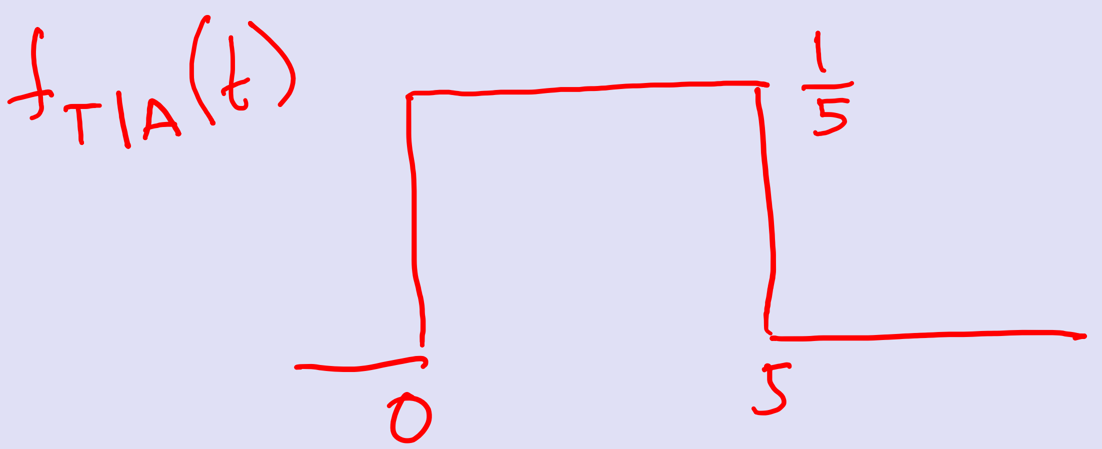


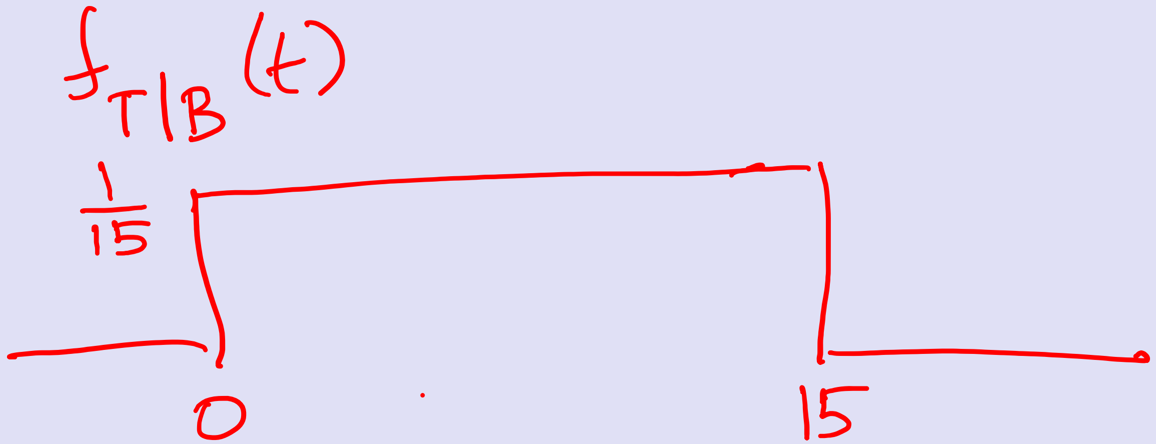
A

B

Arrival time is Uniform(10, 30)

Conditional waiting time PDF:





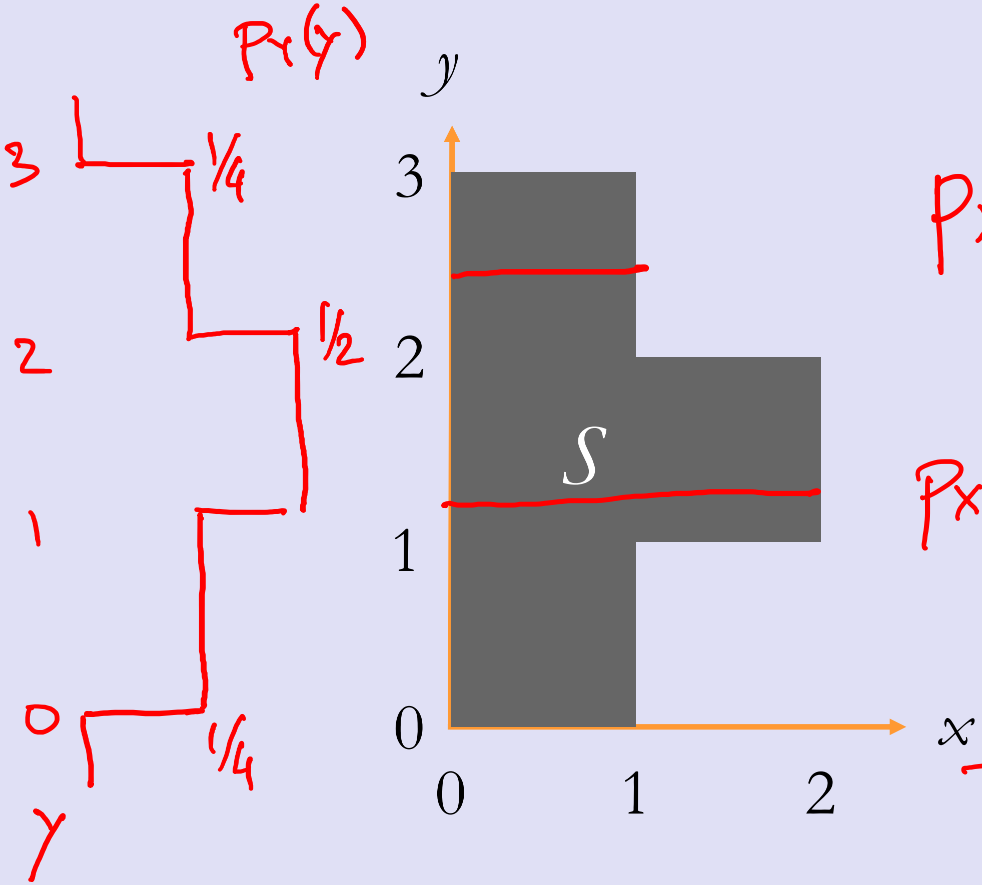
Conditioning

X \ Y	discrete	continuous
discrete	$f_{X\mid Y}(x \mid y) = \dfrac{\overset{\text{PMF}}{f_{XY}(x,y)}}{\underset{\text{PMF}}{f_Y(y)}}$	$f_{X\mid Y}(x\mid y) = \dfrac{\overset{\text{PMF/PDF}}{f_{X,Y}(x,y)}}{\underset{\text{PDF}}{f_Y(y)}}$
continuous	$f_{X\mid Y}(x\mid y) = \dfrac{\overset{\text{PDF/PMF}}{f_{X,Y}(x,y)}}{\underset{\text{PMF}}{f_Y(y)}}$	$f_{X\mid Y}(x\mid y) = \dfrac{\overset{\text{PDF}}{f_{X,Y}(x,y)}}{\underset{\text{PDF}}{f_Y(y)}}$

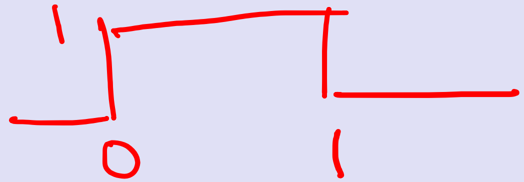
Joint PDF of X, Y is uniform over S.

What are the conditional PDFs?

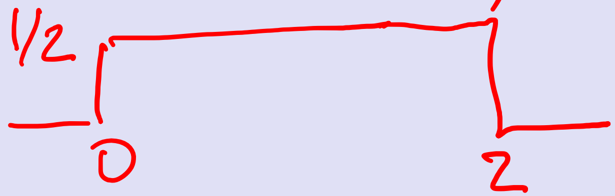
$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x,y)}{p_Y(y)}$$



$$p_{X|Y}(x \mid 2.5) = \text{Uniform}(0,1)$$



$$p_{X|Y}(x \mid 1.2) = \text{Uniform}(0,2)$$

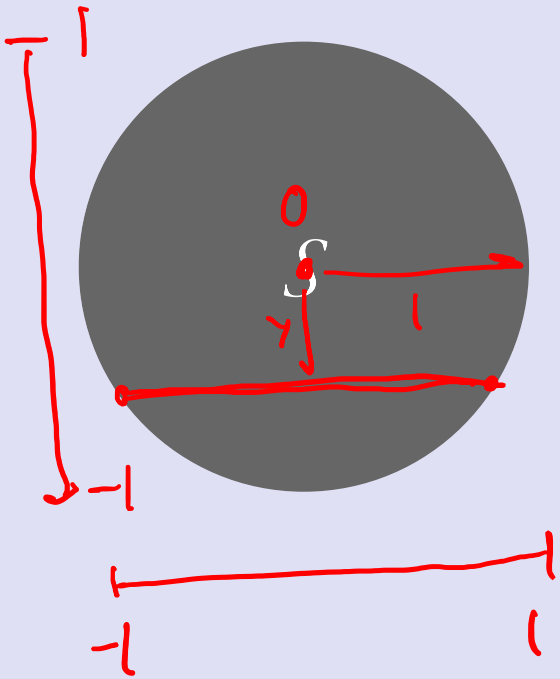


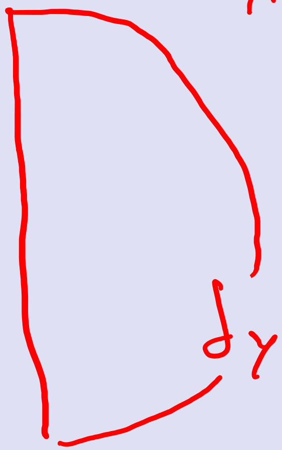
$$p_{X,Y}(x,y) = \frac{1}{4} \text{ WHEN } (x,y) \in S$$

$$p_{X|Y}(x \mid y) = \begin{cases} 1 & \text{WHEN } y \in (0,1) \cup (2,3) \\ 1/2 & \text{WHEN } y \in (1,2) \end{cases}$$

Joint PDF of X, Y is uniform over S.

What are the marginal and conditional PDFs?

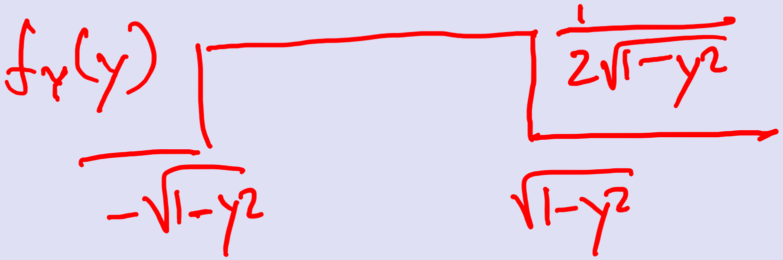




$f_{X,Y}(x,y) = \text{Uniform over } S$

$= \frac{1}{\pi}$ WHEN $(x,y) \in S$

$$f_Y(y) = \int_{x=\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2}{\pi}\sqrt{1-y^2}$$



$$f_{X|Y}(x|y) = \frac{1/\pi}{2/\pi\,\sqrt{1-y^2}}$$

$$= \frac{1}{2\sqrt{1-y^2}}.$$

Total probability theorem:

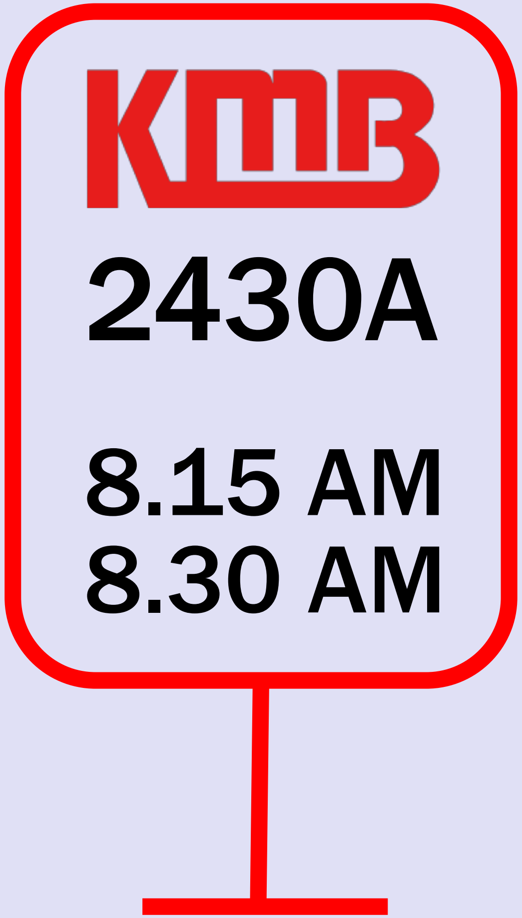
$$f_X(x) = \sum_y f_{X|Y}(x \mid y)\, f_Y(y)$$ Y **discrete**

$$f_X(x) = \int f_{X|Y}(x \mid y)\, f_Y(y)\, dy$$ Y **continuous**

Total expectation theorem:

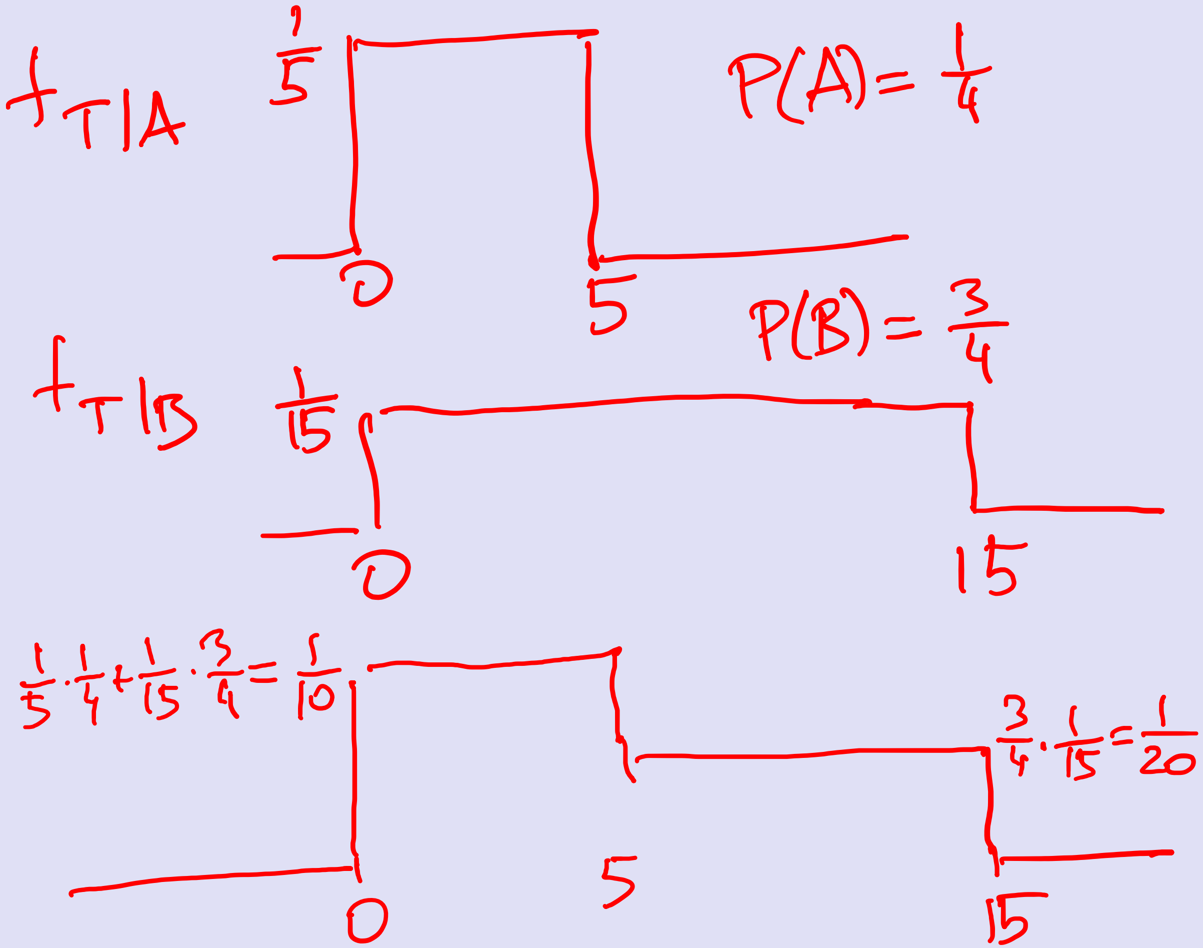
$$\mathbf{E}[X] = \sum_y \mathbf{E}[X \mid Y = y]\, f_Y(y)$$ Y **discrete**

$$\mathbf{E}[X] = \int \mathbf{E}[X \mid Y = y]\, f_Y(y)\, dy$$ Y **continuous**



Arrival time is Uniform(10, 30)

Unconditional waiting time PDF:



f_T

$Y = \text{Normal}(X, X/10)$ $X = \text{Exponential}(1/50)$

$$f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \frac{1}{\sqrt{2\pi \cdot \frac{x}{10}}}\, e^{-\frac{(y-x)^2}{2\cdot(x/10)^2}} \cdot \frac{1}{50} \cdot e^{-\frac{1}{50}x}$$

$$\mathbf{E}[Y] = \int_{-\infty}^{\infty} E[Y|X=x]\, f_X(x)\, dx = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = E[X] = 50$$

Independence

X and Y are independent if

$$f_{XY}(x, y) = f_X(x)\, f_Y(y) \text{ for all } x, y$$

if and only if

$$\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)]\mathbf{E}[h(Y)] \text{ for all } g, h$$

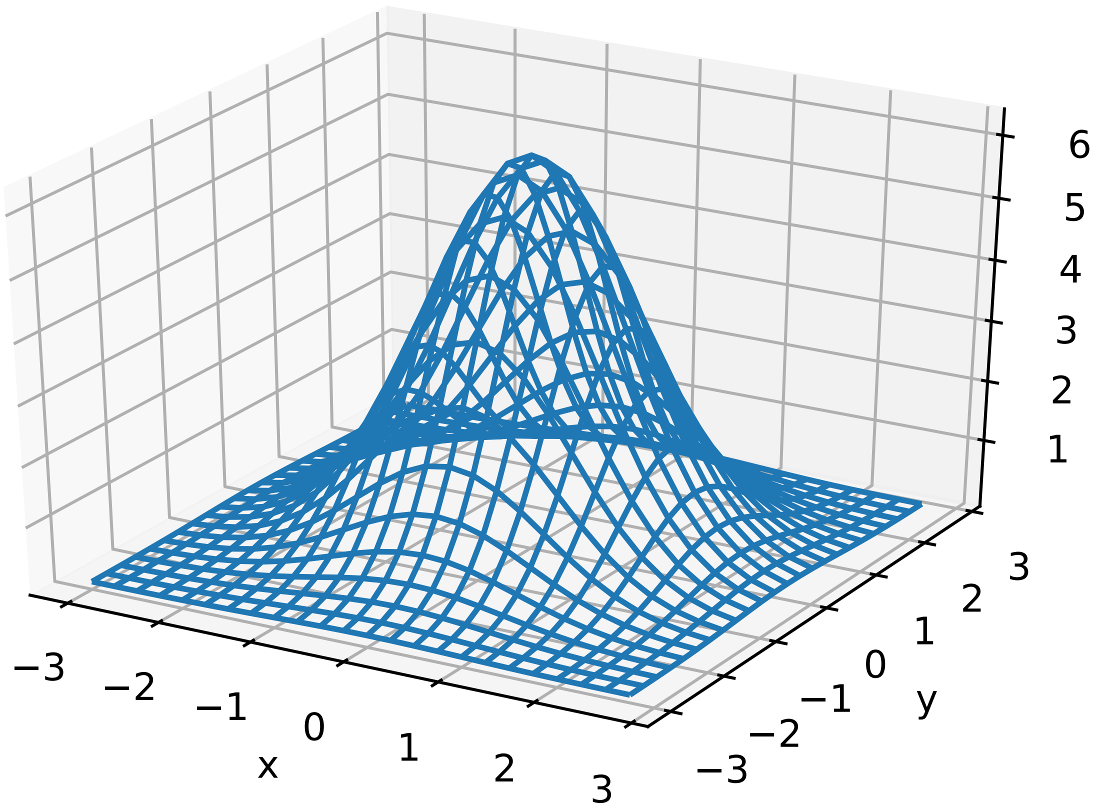
Independent Normals

X, Y **are** $\mathrm{Normal}(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

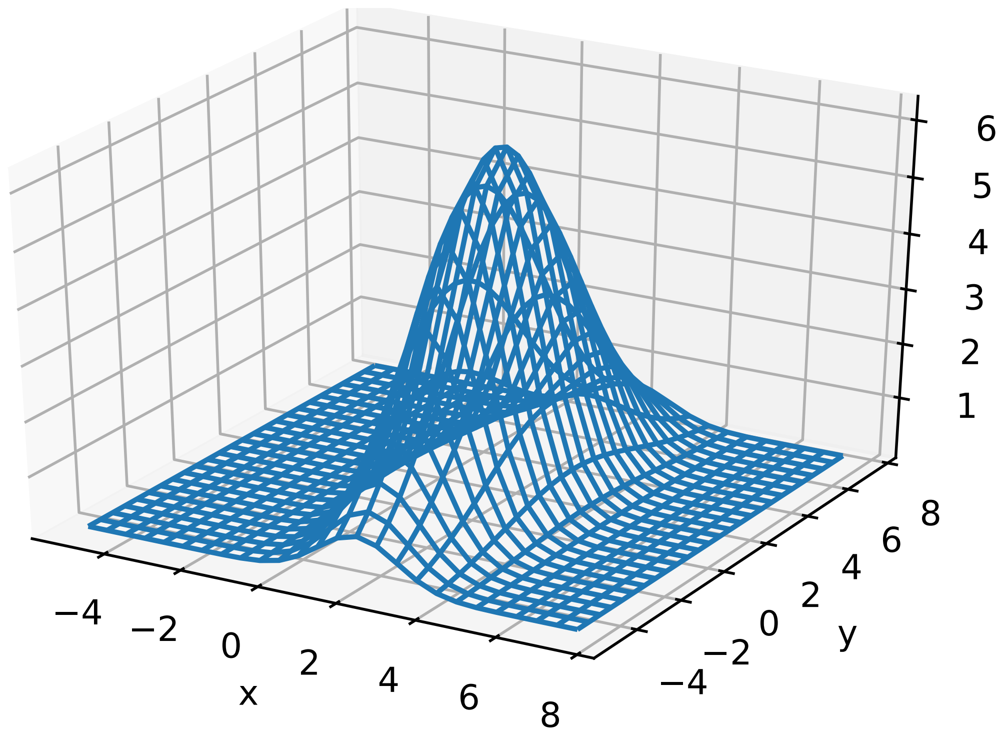
$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

ONLY DEPENDS ON "RADIUS" $\sqrt{x^2+y^2}$



Independent Normals

X **is** $\mathrm{Normal}(\mu, \sigma)$, Y **is** $\mathrm{Normal}(\mu', \sigma')$

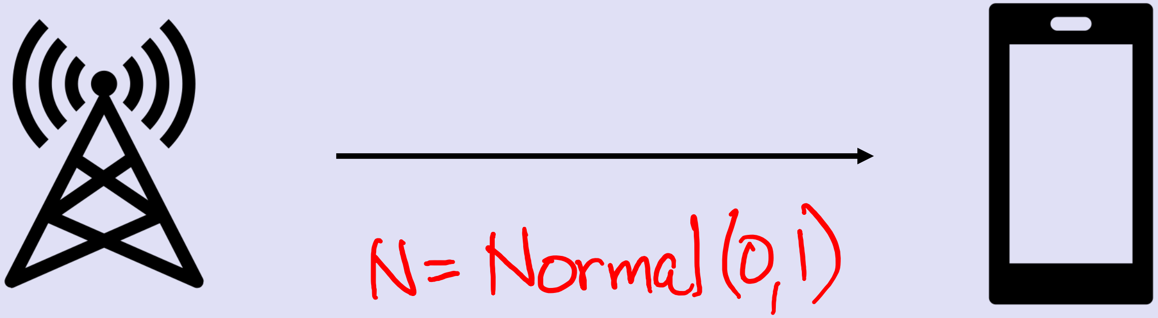


Continous Bayes' rule

$$f_{Y \mid X}(y \mid x) = \frac{f_{X \mid Y}(x \mid y)\, f_Y(y)}{f_X(x)}$$

$$= \frac{f_{X|Y}(x|y)\, f_Y(y)}{\sum_y f_{X|Y}(x|y)\, f_Y(y)} \quad \text{IF } y \text{ DISCRETE}$$

$$= \frac{f_{X|Y}(x|y)\, f_Y(y)}{\int f_{X|Y}(x|y)\, f_Y(y)\, dy} \quad \text{IF } y \text{ CONTINUOUS}$$



$X = 1$ or -1

$\frac{1}{2}$ $\frac{1}{2}$

$Y = X + N$

$$\mathbf{P}(X = 1 \mid Y = y) = \frac{f_Y(y|1)\cancel{P(X=1)}}{f_Y(y|1)\cancel{P(X=1)} + f_Y(y|-1)\cancel{P(X=-1)}}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}}{\frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} + \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2}}$$

$$= \frac{e^{y}}{e^{y}+e^{-y}}$$

Rain falls at $\text{Poisson}(\Lambda)$ **drops/sec**

N

Λ **itself is** $\text{Exponential}(1)$

You're hit by 2 drops. What is your guess for Λ**?**

$$f_{\Lambda|N}(\lambda|2) = \frac{f_{N|\Lambda}(2|\lambda)\, f_\Lambda(\lambda)}{f_N(2)}$$

$$\frac{\lambda^2}{2!}e^{-\lambda}$$

$$e^{-\lambda}$$

$$= \int f_{N|\Lambda}(2|\lambda')\, f_\Lambda(\lambda')\, d\lambda'$$

$$= \frac{\frac{\lambda^2}{2!}\cdot e^{-2\lambda}}{\int_0^\infty \frac{\lambda'^2}{2!}e^{-2\lambda'}\,d\lambda} = \frac{\lambda^2/2!\, e^{-2\lambda}}{1/8} = 4\lambda^2 e^{-2\lambda}$$