ENGG 2430 / ESTR 2004: Probability and Statistics
Spring 2019

2. Conditional Probability

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Coins game

Toss 3 coins. You win if at least two come out heads.

$$S = \{\ \mathrm{HHH},\ \mathrm{HHT},\ \mathrm{HTH},\ \mathrm{HTT},\ \mathrm{THH},\ \mathrm{THT},\ \mathrm{TTH},\ \mathrm{TTT}\ \}$$

$$W = \{\ \mathrm{HHH},\ \mathrm{HHT},\ \mathrm{HTH},\ \mathrm{THH}\ \}$$

$$P(W) = \frac{4}{8} = 50\%$$

Coins game

The first coin was just tossed and it came out heads. How does this affect your chances?

$$S = \{\ \underset{1/4}{\text{HHH}},\ \underset{1/4}{\text{HHT}},\ \underset{1/4}{\text{HTH}},\ \underset{1/4}{\text{HTT}},\ \cancel{\text{THH}, \text{THT}, \text{TTH}, \text{TTT}}\ \}$$

$$W = \{\ \text{HHH}, \text{HHT}, \text{HTH}, \cancel{\text{THH}}\ \}$$

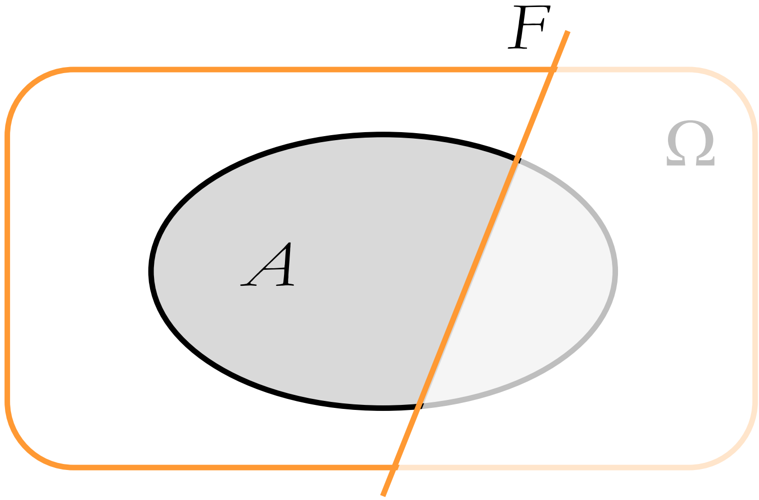
$$P(W|E) = \frac{3}{4} = 75\%$$

$$= \frac{P(W \cap E)}{P(E)} = \frac{3/8}{4/8}$$

Conditional probability

The conditional probability $P(A \mid F)$ represents the probability of event A assuming event F happened.

"A given F"

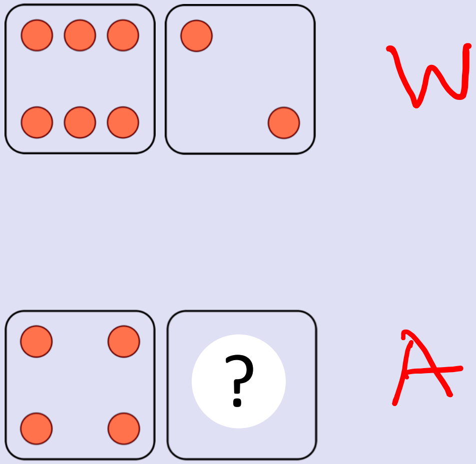


Conditional probabilities with respect to the reduced sample space F are given by the formula

$$\boxed{\mathbf{P}(A \mid F) = \frac{\mathbf{P}(A \cap F)}{\mathbf{P}(F)}}$$

Toss 2 dice. You win if the sum of the outcomes is 8.

The first die toss is a 4. Should you be happy?



$$P(W|A) = \frac{P(W\cap A)}{P(A)} = \frac{1/36}{1/6} = \frac{1}{6}$$

$$P(W) = \frac{5}{36}$$

Now suppose you win if the sum is 7. $\leftarrow$ W'

Your first toss is a 4. Should you be happy?

$\nwarrow A$

$$P(W'|A) = \frac{P(W' \cap A)}{P(A)} = \frac{1/36}{1/6} = \frac{1}{6}$$

$$P(W') = \frac{|W'|}{36} = \frac{6}{36} = \frac{1}{6}$$

Properties of conditional probabilities

1. Conditional probabilities are probabilities:

$$\mathbf{P}(F \mid F) = 1$$

$\mathbf{P}(A \cup B \mid F) = \mathbf{P}(A \mid F) + \mathbf{P}(B \mid F)$ if disjoint

2. Under equally likely outcomes,

$$\mathbf{P}(A \mid F) = \frac{\text{number of outcomes in } A \cap F}{\text{number of outcomes in } F}$$

Toss two dice. The smaller value is a 2. What is the probability that the larger value is 1, 2, ..., 6?

F

E_1 E_6

$$P(E_6|F) = \frac{|E_6 \cap F|}{|F|} = \frac{2}{9}$$

11	12	13	14	~~15~~	~~16~~
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

F

i	$P(E_i\|F)$
6	2/9
5	2/9
4	2/9
3	2/9
2	1/9
1	0

You draw a random card and see a black side.
What are the chances the other side is red?

A: $1/4$ [✓ B: $1/3$] C: $1/2$

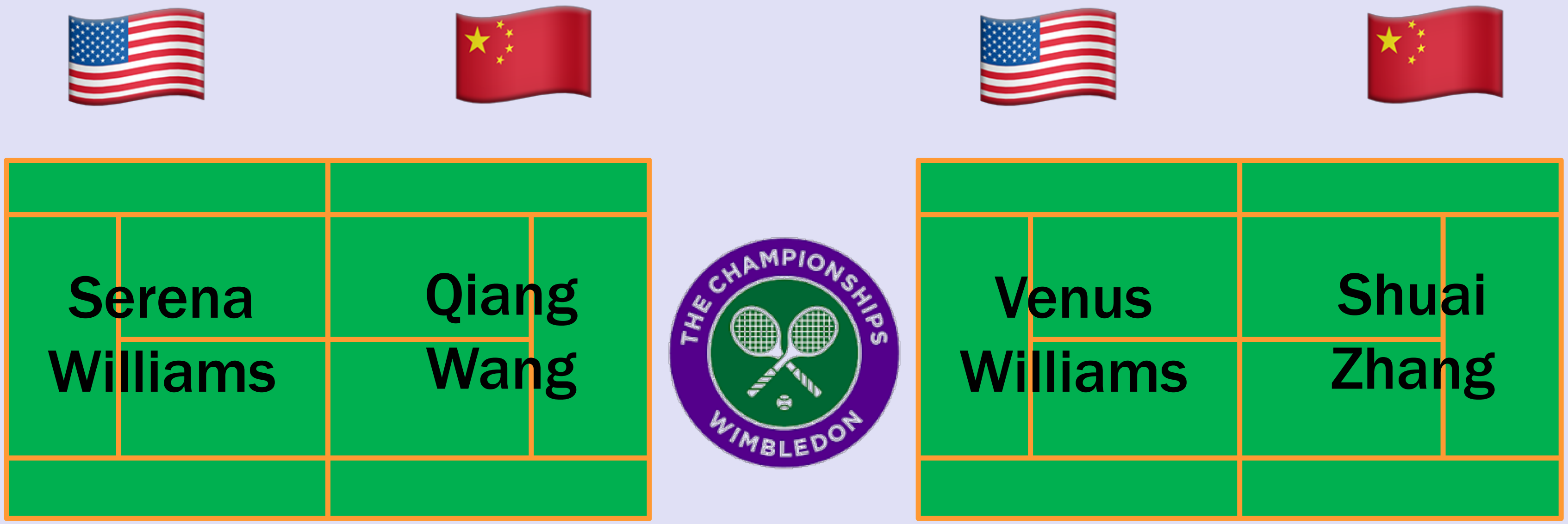
$$\Omega = \{1F, 1B, 2F, 2B, 3F, 3B\}$$

$$E = \{2F, 2B, 3F\}$$

$$F = \{1F, 1B, 3F\}$$

EQUALLY LIKELY OUTCOMES

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{|E \cap F|}{|F|} = \frac{1}{3}$$



🇺🇸 🇨🇳 🇺🇸 🇨🇳

Serena Williams **Qiang Wang**

Venus Williams **Shuai Zhang**

$\mathbf{P}(\text{Serena wins}) = 2/3$

$\mathbf{P}(\text{Venus wins}) = 1/2$

$\mathbf{P}($🇨🇳 $2:$ 🇺🇸 $0) = 1/4$

FINAL SCORE

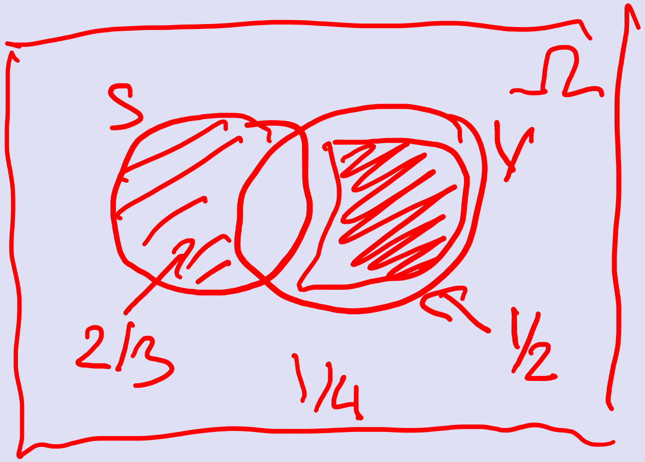
🇨🇳	1
🇺🇸	1

What is the probability Serena won her game?

$$\Omega = \{\underset{\underset{S}{\uparrow}}{W}\underset{\underset{V}{\nwarrow}}{W},\ WL,\ LW,\ LL\}$$

$$S = \{WW, WL\}$$

$$F = \{WL, LW\}$$



$$P(S|F) = \frac{P(S\cap F)}{P(F)} = \frac{P(\{WL\})}{P(F)} = \frac{P(\{WL\})}{P(\{WL\})+P(\{LW\})}$$

$$P(\{WL\}) = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$$

$$P(\{LW\}) = 1 - \frac{1}{4} - \frac{2}{3} = \frac{1}{12}$$

$$P(S|F) = \frac{1/4}{1/4 + 1/12} = \frac{3}{4}$$

The multiplication rule

Using the formula $\mathbf{P}(E_2|E_1) = \dfrac{\mathbf{P}(E_1 \cap E_2)}{\mathbf{P}(E_1)}$

We can calculate the probability of intersection

$$\mathbf{P}(E_1 \cap E_2) = \mathbf{P}(E_1)\ \mathbf{P}(E_2|E_1)$$

In general

$$P(E_1 \cap \ldots \cap E_n) = P(E_1)\ P(E_2|E_1)\ldots \mathrm{P}(E_n|E_1 \cap \ldots \cap E_{n\text{-}1})$$

An urn has 10 white balls and 20 black balls. You draw two at random. What is the probability that both are white?

A = FIRST BALL WHITE

B = 2ND BALL WHITE.

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$= \frac{10}{30} \cdot \frac{9}{29}$$

9W 20B

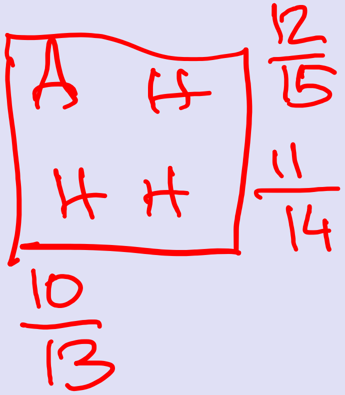
12 HK and 4 mainland students are randomly split into four groups of 4. What is the probability that each group has a mainlander?

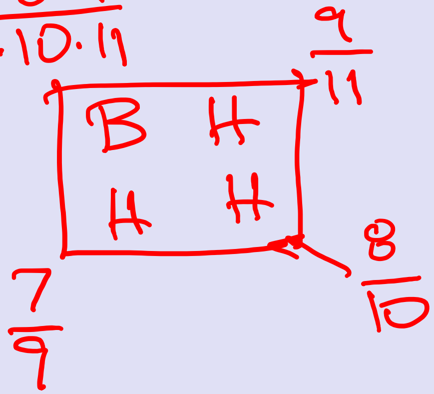
A, B, C, D ML STUDENTS

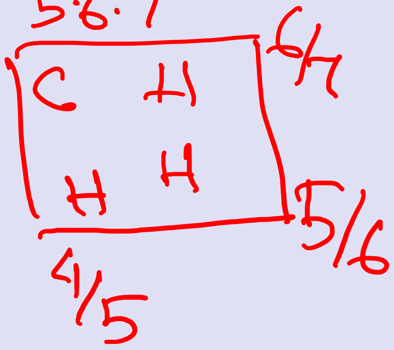
A = STUDENT A IS THE ONLY ML STUDENT IN HIS GROUP.

$$P(A\cap B\cap C\cap D) = P(A)P(B|A)P(C|A\cap B)P(D|A\cap B\cap C)$$

$$P(A) = \frac{10\cdot 11\cdot 12}{13\cdot 14\cdot 15} \qquad P(B|A) = \frac{7\cdot 8\cdot 9}{9\cdot 10\cdot 11} \qquad P(C|A\cap B) = \frac{4\cdot 5\cdot 6}{5\cdot 6\cdot 7} \qquad P(D|A\cap B\cap C) = 1$$



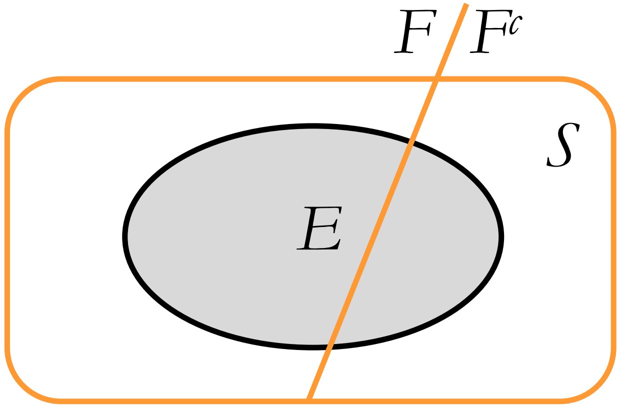




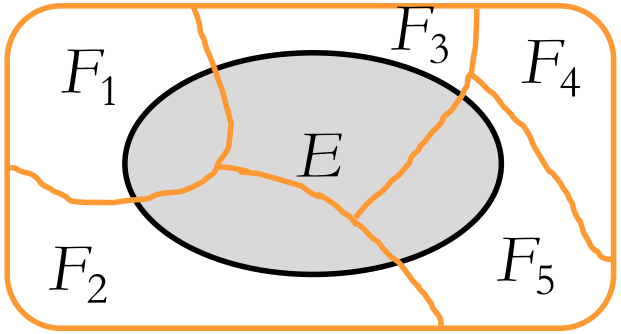
Total probability theorem

$$\mathbf{P}(E) = \mathbf{P}(E \cap F) + \mathbf{P}(E \cap F^c)$$

$$= \mathbf{P}(E \mid F)\mathbf{P}(F) + \mathbf{P}(E \mid F^c)\mathbf{P}(F^c)$$

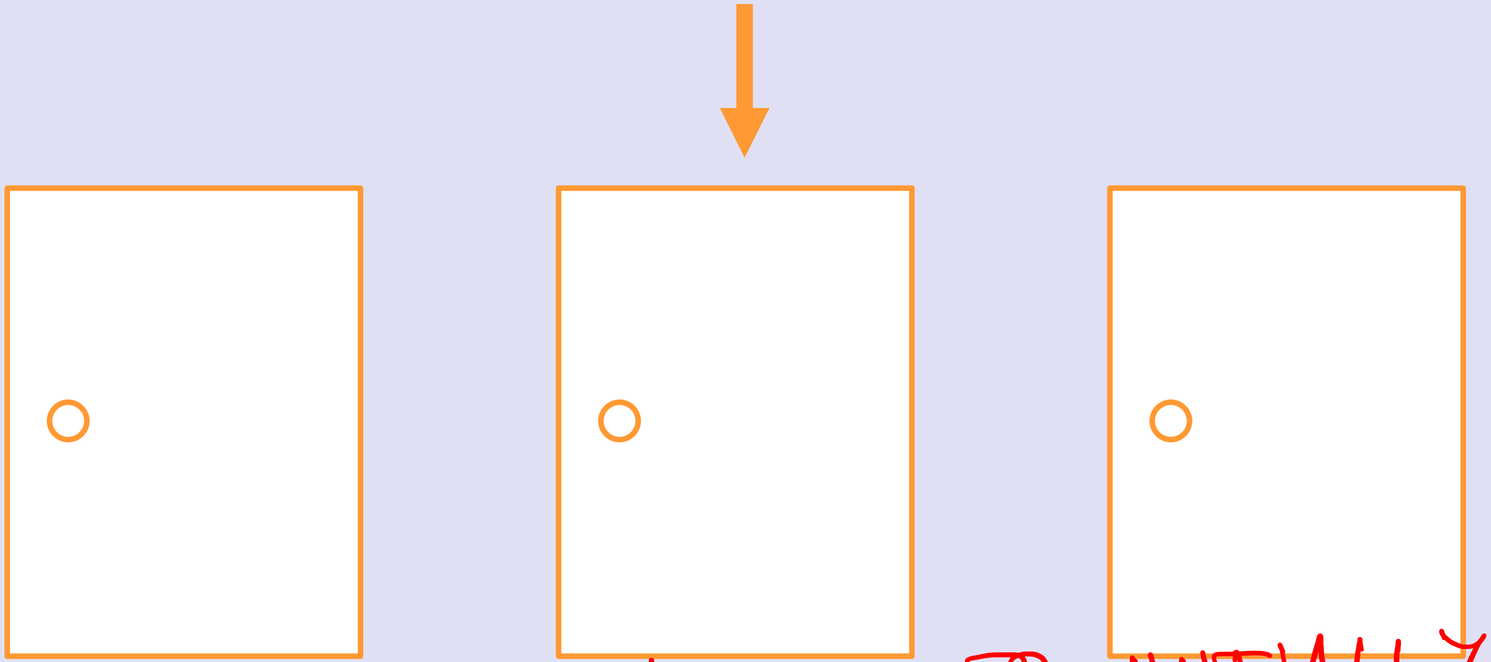


More generally, if $F_1, \dots, F_n$ partition Ω then



$$\mathbf{P}(E) = \mathbf{P}(E \mid F_1)\mathbf{P}(F_1) + \dots + \mathbf{P}(E \mid F_n)\mathbf{P}(F_n)$$

An urn has 10 white balls and 20 black balls. You draw two at random. What is the probability that their colors are different?



A = I PICKED A WINNER INITIALLY

W = I WIN THE PRIZE:

STRATEGY: SWITCH

$$P(W) = \underset{0}{\underline{P(W|A)}}\,\underset{1/3}{\underline{P(A)}} + \underset{1}{\underline{P(W|A^c)}}\,\underset{2/3}{\underline{P(A^c)}} = \frac{2}{3}$$

Multiple choice quiz

What is the capital of Macedonia?

A: Split

B: Struga

20%

[✓ C: Skopje]

50%

D: Sendai

30%

Did you know or were you lucky?

Multiple choice quiz

Probability model

There are two types of students:

Type K: Knows the answer

Type K^c: Picks a random answer

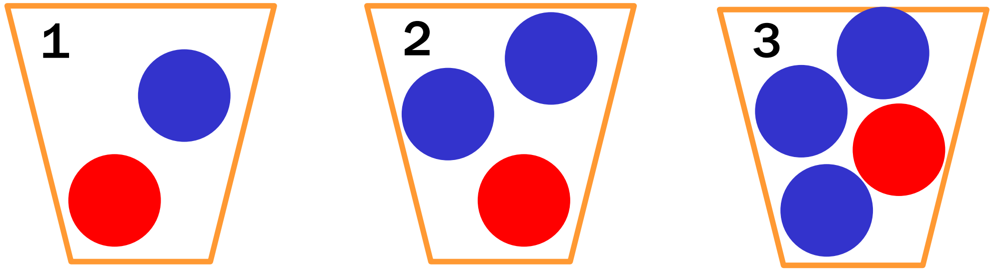
Event C: Student gives correct answer

$P(C) = p =$ fraction of correct answers

$$p = \underset{\substack{\downarrow \\ 1}}{P(C \mid K)}\,\boxed{P(K)} + \underset{\substack{\downarrow \\ 1/4}}{P(C \mid K^c)}\,\underset{\substack{\downarrow \\ 1 - P(K)}}{P(K^c)} = 1/4 + 3P(K)/4$$

$$\boxed{P(K) = (p - \tfrac{1}{4}) \,/\, \tfrac{3}{4}}$$

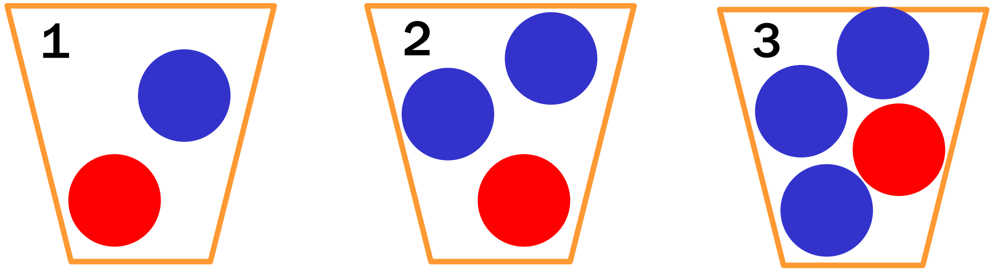
$p = 50\%$ $\qquad$ $P(K) \approx 33\%$



I choose a cup at random and then a random ball from that cup. The ball is red. You need to guess where the ball came from.

Which cup would you guess?

Cause and effect



cause: C_1 C_2 C_3

effect: R

$P(C_i | R)$

INFO: $P(R|C_i)$

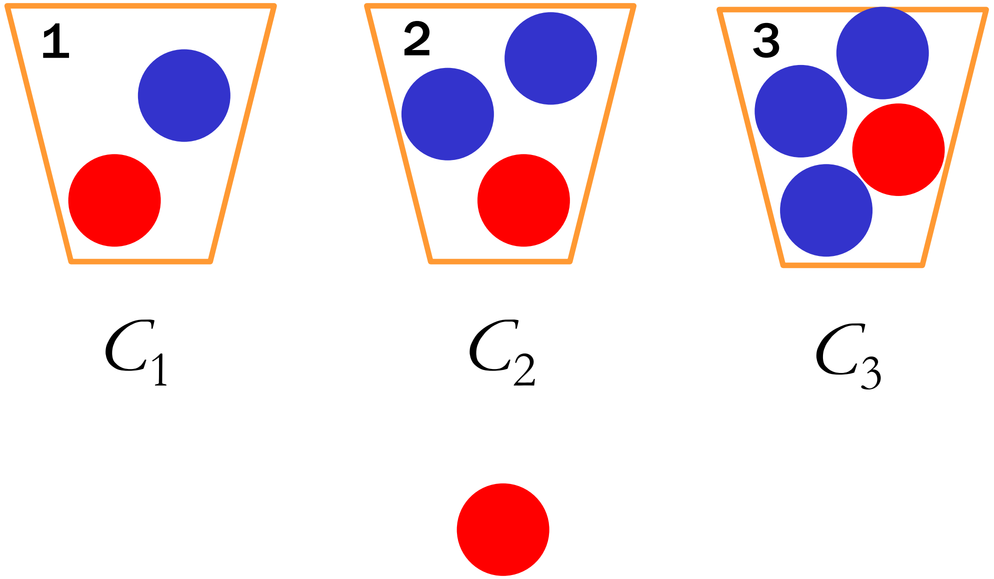
Bayes' rule

$$\mathbf{P}(C \mid E) = \frac{\mathbf{P}(E \mid C)\ \mathbf{P}(C)}{\mathbf{P}(E)} = \frac{\mathbf{P}(E \mid C)\ \mathbf{P}(C)}{\mathbf{P}(E \mid C)\ \mathbf{P}(C) + \mathbf{P}(E \mid C^c)\ \mathbf{P}(C^c)}$$

More generally, if $C_1, \ldots, C_n$ partition S then

$$\mathbf{P}(C_i \mid E) = \frac{\mathbf{P}(E \mid C_i)\ \mathbf{P}(C_i)}{\mathbf{P}(E \mid C_1)\ \mathbf{P}(C_1) + \ldots + \mathbf{P}(E \mid C_n)\ \mathbf{P}(C_n)}$$

Cause and effect

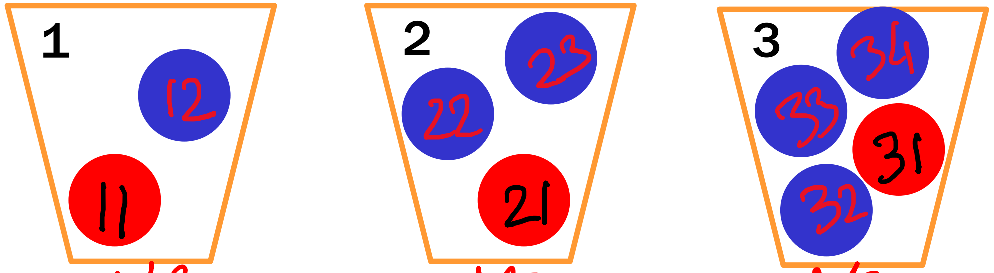


cause: C_1 C_2 C_3

effect: R

$$\underline{\mathbf{P}(C_i\,|\,R)} = \frac{\mathbf{P}(R\,|\,C_i)\;\overset{1/3}{\boxed{\mathbf{P}(C_i)}}}{\mathbf{P}(R\,|\,C_1)\;\underset{1/3}{\boxed{\mathbf{P}(C_1)}} + \mathbf{P}(R\,|\,C_2)\;\underset{1/3}{\boxed{\mathbf{P}(C_2)}} + \mathbf{P}(R\,|\,C_3)\;\underset{1/3}{\boxed{\mathbf{P}(C_3)}}}$$

Cause and effect



1/3 1/3 1/3

$$\Omega = \{\overset{1/6}{11}, \overset{1/6}{12}, \overset{1/9}{21}, \overset{1/9}{22}, \overset{1/9}{23}, \overset{1/12}{31}, \overset{1/12}{32}, \overset{1/12}{33}, \overset{1/12}{34}\}$$

$$\mathbf{P}(C_i) = \quad 1/3 \quad 1/3 \quad 1/3$$

$$\mathbf{P}(R \mid C_i) = \quad 1/2 \quad 1/3 \quad 1/4$$

$$P(C_i|R) = \quad \frac{1/2}{1/2 + 1/3 + 1/4} \quad \frac{1/3}{1/2+1/3+1/4} \quad \frac{1/4}{1/2+1/3+1/4}$$

Two classes take place in Lady Shaw Building.

ENGG2430 has 100 students, 20% are girls.

NURS2400 has 10 students, 80% are girls.

A girl walks out. What are the chances that she is from the engineering class?

CAUSE: E E^c

EFFECT: G

$P(E|G)$

$$P(E|G) = \frac{\overset{\frac{20}{100}}{P(G|E)}\,P(E) \xrightarrow{} \frac{100}{110}}{\underset{\frac{20}{100}}{P(G|E)}\,\underset{\frac{100}{110}}{P(E)} + \underset{\frac{80}{100}}{P(G|E^c)}\,\underset{\frac{10}{110}}{P(E^c)}} = \frac{5}{7}$$

Summary of conditional probability

Conditional probabilities are used:

① When there are causes and effects

to estimate the probability of a cause when we observe an effect

② To calculate ordinary probabilities

Conditioning on the right event can simplify the description of the sample space