Practice questions

1. The PDF of the random variable X is $f(x) = Ce^{-|x|}$, where x ranges over all real numbers. Determine (a) the value of C, (b) the CDF $\mathrm{P}(X \leq x)$, and (c) the probability that $|X| \leq 1$.

   Solution:

   (a) The PDF must integrate to one, so $1 = \int_{-\infty}^{+\infty} f_X(x)dx = \int_{-\infty}^{+\infty} Ce^{-|x|}dx = 2C$. Therefore $C = 1/2$.

   (b) $F_X(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)dx = \int_{-\infty}^{x} \frac{1}{2}e^x dx = \frac{1}{2}e^x$ when $x \leq 0$, and
   $F_X(x) = \mathrm{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)dx = \int_{-\infty}^{0} \frac{1}{2}e^x dx + \int_0^x \frac{1}{2}e^{-x}dx = 1 - \frac{1}{2}e^{-x}$ when $x > 0$.
   Therefore,
   $$F_X(x) = \begin{cases} \frac{1}{2}e^x, & \text{if } x \leq 0, \\ 1 - \frac{1}{2}e^{-x}, & \text{otherwise.} \end{cases}$$

   (c) $\mathrm{P}(|X| \leq 1) = F_X(1) - F_X(-1) = 1 - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-1} = 1 - e^{-1}$.

2. The joint PDF of (X, Y) is
   $$f_{X,Y}(x, y) = \begin{cases} C(x + y + 1)y, & \text{if } 0 \leq x \leq 2, 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$
   Find (a) the value of C and (b) The conditional PDF $f_{Y|X}(y|x)$.

   Solution:

   (a) The PDF must integrate to one, so
   $$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x, y)dxdy = \int_0^2\int_0^2 C(x + y + 1)ydxdy = \frac{40}{3}C.$$
   Therefore $C = \frac{3}{40}$.

   (b) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dy = \int_0^2 C(x + y + 1)ydy = C(2x + \frac{14}{3})$. Using the convolution formula,
   $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{(x + y + 1)y}{2x + \frac{14}{3}}.$$

3. Let X be a Uniform$(0, 1)$ random variable. Find the PDF of the random variables (a) $Y = e^X$ and (b) $Z = -2 \log X$.

   Solution:

   (a) The CDF of Y is $F_Y(y) = P(e^X \leq y) = P(X \leq \log y) = \log y$.
   By differentiating, we have
   $$f_Y(y) = \begin{cases} 1/y, & \text{if } 1 \leq y \leq e, \\ 0, & \text{otherwise.} \end{cases}$$

(b) The CDF of Y is $F_Y(y) = P(-2\log X \le y) = P(X \ge e^{-y/2}) = 1 - e^{-y/2}$.
By differentiating, we have

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2}, & \text{if } y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

4. Raindrops hit your head at a rate of 1 per second. What is the PDF of the time at which the second raindrop hits you? How about the third one? (**Hint:** convolution)

Solution:

The time before the second raindrop is $Y = X_1 + X_2$, where X_1 and X_2 are independent Exponential(1) random variables. We calculate the PDF of A using the convolution formula:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1 = \int_0^y e^{-x_1} e^{-y+x_1} dx_1 = ye^{-y}.$$

The third raindrop hits at time $Z = Y + X_3$, where X_3 is another independent Exponential(1) random variable. By the convolution formula again,

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_{X_3}(z - y) dy = \int_0^z ye^{-y} e^{-z+y} dy = \frac{z^2}{2} e^{-z}.$$

5. You draw 10 balls at random among 15 red and 5 blue balls. Let X be the number of red balls drawn.

(a) What is the expected value of X?

(b) Write $X = X_1 + X_2 + \cdots + X_{10}$, where X_i indicates if the i-th drawn ball is red. What is the variance of X_i?

(c) What is the covariance of X_i and X_j $(i \ne j)$?

(d) What is the variance of X?

Solution:

(a) Let $X = X_1 + X_2 + \cdots + X_{10}$, where X_i indicates if the i-th drawn ball is red. By linearity of expectation,

$$\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{10}] = 10 \cdot \frac{15}{20} = 7.5.$$

(b) $\mathrm{Var}[X_i] = \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2 = \mathrm{P}(X_i = 1) - \mathrm{P}(X_i = 1)^2 = \frac{3}{4} - (\frac{3}{4})^2 = \frac{3}{16}$.

(c) $\mathrm{Cov}[X_i, X_j] = \mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\,\mathrm{E}[X_j] = \mathrm{P}(X_i = 1, X_j = 1) - \mathrm{P}(X_i = 1)\,\mathrm{P}(X_j = 1) = \frac{15}{20} \cdot \frac{14}{19} - (\frac{15}{20})^2 = -\frac{3}{304}$. The variables X_i and X_j are negatively correlated: Given that ball i is red, ball j is less likely to be red.

(d) The variance of X is the sum of the 10 variances from part (b) plus the $10 \cdot 9$ covariances from part (c), so $\mathrm{Var}[X] = 10 \cdot \frac{3}{16} + 10 \cdot 9 \cdot \frac{-3}{304} = 0.9868$.