1. The joint probability density function of the lifetimes X and Y of two connected components in a machine is

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & x \geq 0, y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the probability that the lifetime X of the first component exceeds 3?

   Solution: $\mathrm{P}(X > 3) = \int_3^\infty \int_0^\infty xe^{x(1-y)} dydx = \int_3^\infty e^{-x} dx = e^{-x}|_3^\infty = 0.05$.

   (b) Are X and Y independent? Justify your answer.

   Solution: No. If X and Y were independent then $f_{X,Y}(1,1)f_{X,Y}(2,2) = 2e^{-8}$ and $f_{X,Y}(1,2)f_{X,Y}(2,1) = 2e^{-7}$ should both be equal to the same value $f_X(1)f_X(2)f_Y(1)f_Y(2)$. Alternatively one can calculate the marginal PDFs

$$f_X(x) = \int_0^\infty xe^{-x(1+y)} dy = e^{-x},$$
$$f_Y(y) = \int_0^\infty xe^{-x(1+y)} dx = \frac{1}{(1+y)^2}$$

   for $x, y \geq 0$ and conclude that $f_X(x)f_Y(y)$ is not the same function as $f_{X,Y}(x,y)$.

2. A radio station gives a gift to the third caller who knows the birthday of the radio talk show host. Each caller has a 0.7 probability of guessing the host's birthday, independently of other callers.

   (a) What is the probability mass function of the number of calls necessary to find the winner?

   Solution: n calls are necessary to find the winner if the n-th guess is correct and there are exactly two correct guesses among the first $n-1$. The probability of this is

$$\mathrm{P}(N = n) = \binom{n-1}{2} \cdot 0.7^3 \cdot 0.3^{n-3}.$$

   (b) What is the probability that the station will need five or more calls to find a winner?

   Solution: No winner has been found in the first four calls if the number of correct guesses in those calls is 0, 1, or 2. The probability of this is

$$\mathrm{P}(N \geq 5) = 0.3^4 + 4 \cdot 0.7 \cdot 0.3^3 + \binom{4}{2} \cdot 0.7^2 \cdot 0.3^2 = 0.3483.$$

3. Alice sends a message a that equals -1 or 1. Bob receives the value B which is a Normal random variable with mean a and standard deviation 0.5. Bob guesses that Alice sent 1 if $B > 0.5$, that Alice sent -1 if $B < -0.5$, and declares failure otherwise (when $|B| \leq 0.5$).

(a) What is the probability that Bob declares failure?

Solution: For either message, failure occurs when a normal random variable is between 1 and 3 standard deviations from the mean on one side. If N is a Normal$(0,1)$ random variable then

$$\mathrm{P}(1 \le N < 3) = \mathrm{P}(N < 3) - \mathrm{P}(N < 1) \approx 0.9987 - 0.8413 = 0.1574.$$

(b) Given that Bob didn't declare failure, what is the probability that his guess is correct?

Solution: By symmetry we may assume Alice sent 1. The event "Bob's guess is correct *and* failure didn't occur" happens when B takes value 0.5 or larger, or when a Normal$(0,1)$ random variable N takes value at most 1, which is approximately 0.8413. Therefore the conditional probability that Bob's guess is correct is about $0.8413/(1-0.1574) \approx 0.9985$.

4. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. There are 20 floors above (not including) the ground floor and each person is equally likely to get off on any one of them, independently of all others.

(a) What is the probability p that the elevator doesn't stop on the seventh floor?

Solution: The number of passengers bound for the seventh floor is a Poisson random variable with mean $10/20 = 1/2$, so the probability that no passengers land there is the probability that this random variable takes value zero, which is $p = e^{-1/2} \approx 0.6065$.

Alternatively, by conditioning on the number N of passengers who enter the elevator and applying the total probability theorem,

$$\begin{aligned} p = \sum_{n=0}^{\infty} (19/20)^n \cdot \mathrm{P}(N = n) &= \sum_{n=0}^{\infty} (19/20)^n \cdot e^{-10} \cdot \frac{10^n}{n!} \\ &= e^{-10} \cdot \sum_{n=0}^{\infty} \frac{(10 \cdot 19/20)^n}{n!} = e^{-10} \cdot e^{10 \cdot 19/20} = e^{-1/2}. \end{aligned}$$

(b) What is the expected number of stops that the elevator will make?
(Express the answer in terms of p in case you didn't complete part (a).)

Solution: By linearity of expectation, the expected number of stops is the sum of the probabilities that the elevator stops on floor 1 up to floor 20. By part (a) each of these probabilities is $1 - e^{-1/2}$ so the answer is $20 \cdot (1 - e^{-1/2}) \approx 7.869$.

5. 500 balls are drawn without replacement from a bin with 600 black balls and 400 white balls.

   (a) What is the expected number of black balls drawn?

   Solution: In any given draw the probability that the ball is black is $600/1000 = 3/5$. By linearity of expectation the expected number of black balls drawn is $(3/5) \cdot 500 = 300$.

   (b) What is the variance of the number of black balls drawn?

   Solution: The variance of the number of black balls is the sum of the 500 variances v indicating that any given ball is black plus the $500 \cdot 499$ covariances c indicating that any two given balls are black. Each individual variance is $v = 3/5 \cdot 2/5 = 6/25$, while each of the covariances equals

$$c = \frac{600}{1000} \cdot \frac{599}{999} - \left(\frac{600}{1000}\right)^2 \approx -2.402 \cdot 10^{-4},$$

   from where the variance is $500v + 500 \cdot 499 \cdot c \approx 60.06$.

   (c) Is the probability you drew fewer than 200 black balls more than 2%? Justify your answer.

   Solution: The standard deviation of the number of black balls drawn is about $\sqrt{60.06} \approx 7.750$. In order to draw fewer than 200 black balls, the number of black balls drawn must be $100/7.750 \approx 12.90$ standard deviations lower than the mean. By Chebyshev's inequality, the probability of this is at most $1/12.90^2 \approx 0.006$. This is much less than 2%.

   (**Note:** Even if the covariance term in part (b) is mistakenly omitted, the calculation comes out to less than 2%.)