ENGG 2760A / ESTR 2018: Probability for Engineers

Tutorial 6

1. Quiz 4
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3. Homework 7

Quiz 4:

Eight boys and eight girls are randomly seated at a round table. What is the expected number of boys that are seated between two girls?

1) $E[X] = E[X_1] + E[X_2] + \cdots + E[X_{16}]$

X_i Girl Boy Girl $P(X_i) = \frac{1}{2} \times \frac{8}{15} \times \frac{7}{14}$

i-th seat

$E[Y] = E[Y_1] + \cdots E[Y_8]$

Y_i: ith boy seated 2 girls

2) $E[X] = \sum_{i=0}^{8} i P(X=i)$

linearity of expectation

indicator variable $X_i = \begin{cases} 1 \\ 0 \end{cases}$

3) GBBBG X

Quiz 5:

A five person committee is chosen at random from 5 girls and 15 boys. Conditioned on there being at least one girl on the committee, what is the expected number of boys?

20

A^c

A: 0 girl on the committee

$E[X_i] = p(X_i)$

$E[\text{girls}] = E[\text{girls} \mid A]\, p(A) + E[\text{girls} \mid A^c]\, p(A^c)$

$= E[X_1] + \cdots + E[X_5]$ (X_i: i-th girl)

$\frac{5}{20} \times 5 = 0 \times \frac{\binom{15}{5}}{\binom{20}{5}} + E[\text{girls} \mid A^c]\left(1 - \frac{\binom{15}{5}}{\binom{20}{5}}\right)$

$E[\text{girls} \mid A^c] \approx 1.55$

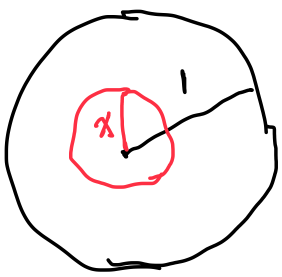
boys = 5 − girls

$E[\text{boys} \mid A^c] = 5 - E[\text{girls} \mid A^c] = 5 - 1.55 = 3.45$

HW7

Q1. (a) CDF: $P(X \le x) = F(x) = \dfrac{\text{Area}(x)}{\text{Area}(1)} = \dfrac{\pi x^2}{\pi} = x^2$

$x \in [0,1]$



(b) PDF: $f(x) = \dfrac{dx^2}{dx} = 2x$

$$F(x) = \begin{cases} x^2 & x \in [0,1] \\ 1 & x \in (1, +\infty) \end{cases}$$

$$f(x) = \begin{cases} 2x & \boxed{x \in [0,1]} \\ 0 & x \in (1, +\infty) \end{cases}$$

(c) $E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}$

(d) $Var(X) = E[X^2] - (E[X])^2$

$= \int_{-\infty}^{+\infty} x^2 f(x)\,dx - \left(\frac{2}{3}\right)^2 = \int_{-\infty}^{+\infty} 2x^3\,dx - \frac{4}{9}$

$= \frac{2}{4}x^4 \Big|_0^1 - \frac{4}{9} = \frac{1}{18}$

CDF|condition → PDF|condition → E[X|condition]

2. (e) $P(X \le x \mid X > 0.5) = \dfrac{P(0.5 < X \le x)}{P(X > 0.5)} = \dfrac{F(x) - F(0.5)}{1 - \underbrace{P(X \le 0.5)}_{F(0.5)}} = \dfrac{x^2 - 0.25}{0.75}$

$x \in [0.5, 1]$

$F(x) = x^2$

PDF $f(x \mid x > 0.5) = \dfrac{dF(x \mid x > 0.5)}{dx} = \begin{cases} \frac{8}{3}x & x \in [0.5, 1] \\ 0 & o/w \end{cases}$

$E[X \mid X > 0.5] = \int_{0.5}^{1} \frac{8}{3}x^2\,dx = \frac{8}{9}x^3 \Big|_{0.5}^{1} = \frac{7}{9} = \int_{-\infty}^{+\infty} x P(x \mid x > 0.5)\,dx$

3.(a) ← 1 →

12:00 $3/n$ 13:00 $1/n$ 14:00 $1/n$ ---

$I_n/n < 1$ $\quad 1 < I_n/n < 2$

i-th interval

$$P(E_i) = \begin{cases} \frac{3}{n} & 1 \le i \le n \\ \frac{1}{n} & i > n \end{cases}$$

(b) the first bus arrives at t-th interval $\quad$ Geometric variables

$$P(I_n = t) = \begin{cases} \left(1-\frac{3}{n}\right)^{t-1} \cdot \frac{3}{n} & i \in [1, n] \\ \left(1-\frac{3}{n}\right)^{n} \left(1-\frac{1}{n}\right)^{t-n-1} \cdot \frac{1}{n} & i \in (n, +\infty) \end{cases}$$

PDF

in first n-th intervals $\quad$ t-n-1 no bus $\quad$ in t

CDF of I_n

$P(I_n \le t) = 1 - P(I_n > t)$ ← in 1~t-th interval no buy arrives

$$= \begin{cases} 1-\left(1-\frac{3}{n}\right)^{t} & i \in [1, n] \\ \left(1-\frac{3}{n}\right)^{n}\left(1-\left(1-\frac{1}{n}\right)^{t-n}\right) & i \in (n, +\infty) \end{cases}$$

T: which hour

(c) $F_T(t) = \lim\limits_{n\to\infty} P\left(\boxed{I_n/n} \le t\right) = \lim\limits_{n\to\infty} P(I_n \le nt)$

$$\lim_{n\to\infty}\left(1-\frac{a}{n}\right)^{n} = e^{-a}$$

$$F_T(t) = \begin{cases} \lim\limits_{n\to\infty} \left[1 - \left(1-\frac{3}{n}\right)^{nt}\right] \\ \lim\limits_{n\to\infty} \left(1-\frac{3}{n}\right)^{n} \left(1 - \left(1-\frac{1}{n}\right)^{nt-n}\right) \end{cases}$$

$\uparrow$ CDF

$$= \begin{cases} 1 - e^{-3t} & t \in [0,1] \\ e^{-3}\left(1 - e^{-(t-1)}\right) & t \in (1,\infty) \end{cases}$$

(e)

$$\text{PDF } f_T(t) = \frac{dF_T(t)}{dt} = \begin{cases} 3e^{-3t} & t \in [0.1] \\ e^{-3-(t-1)} = e^{-2-t} & t \in (1,+\infty) \end{cases}$$

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t)\,dt = \int_0^1 3te^{-3t}\,dt + \int_1^{\infty} te^{-2-t}\,dt$$

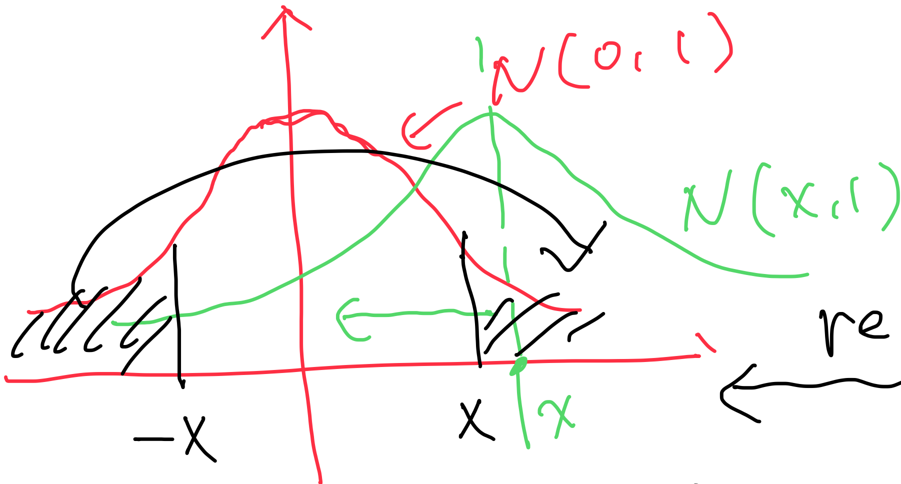
$$= \frac{2}{3}e^{-3}$$

4. decoding is wrong

$$E[Y] = (X+10)\Big[\underbrace{P(Y\le 0 \mid m=1)}_{N(x,1)} + \underbrace{P(Y>0 \mid m=-1)}_{N(-x,1)}\Big]$$

$$+ X\left[P(Y>0 \mid m=1) + P(Y\le 0 \mid m=-1)\right]$$

$N(x,1) \rightarrow N(0,1)$



$\Phi(x) = P(X \le x)$

red line $\leftarrow$ $\boxed{N(0,1)}$

$\Phi(-x) + \Phi(x) = 1$

$$E[Y] = (x+10)\left(\Phi(-x) + (1-\Phi(x))\right)$$
$$+ x\left(1 - \Phi(-x) + \Phi(x)\right)$$

$$= (x+10)\,2(1-\Phi(x)) + 2x\Phi(x)$$

$$= 2x + 20 - 20\Phi(x) = g(x)$$

$$g'(x) = 2 - 20\Phi'(x) = 2 - 20 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0$$

$\uparrow$ PDF of standard Normal V

$$\Rightarrow x = \sqrt{2\ln\frac{10}{\sqrt{2\pi}}}$$