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Spring 2022 Scribe: Siu On Chan

Notes 04: Conjugate function

1. CONVEX FUNCTIONS

Definition 1.1. A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ on *n*-dimensional Euclidean space is convex if for every $x, y \in \mathbb{R}^n$ and every $0 \leq \lambda \leq 1$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

In other words, if we consider the graph of a function, defined as $\{(x, f(x)) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}$, then f is convex if the line segment between any two points of the graph lies above or on the graph.

2. Conjugate function

We now define a dual object for every function $f: \mathbb{R}^n \to \mathbb{R}$, called its conjugate.

We have defined dual objects for sets, using support functions. To define a dual object for a function, we want to first turn $f : \mathbb{R}^n \to \mathbb{R}$ into a set.

Given a function $f: \mathbb{R}^n \to \mathbb{R}$ (not necessarily convex), its epigraph is epi $f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid$ $f(x) \leqslant t\}.$

Note that a function is convex if and only if its epigraph is a convex set, as can be easily checked. The conjugate of a function f is essentially the support function of epi f, "simplified".

The support function of epi f is $S_{\text{epi } f}(y, s) = \sup\{\langle y, x \rangle + st \mid x \in \mathbb{R}^n, f(x) \leq t\}.$

But if s > 0, $S_{\text{epi}f}$ says nothing about f, because the supremum is $+\infty$ by taking arbitrarily large t. If s = 0, S_{epif} also says nothing about f. Only when s < 0 does S_{epif} capture information about f. In this case we always choose t = f(x) in the supremum without changing the outcome. Given any (y, s) with s < 0, we can renormalize (y, s) so that s = -1. This motivates the following definition.

Definition 2.1. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, its conjugate $f^* : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$f^*(y) = \sup\{\langle y, x \rangle - f(x) \mid x \in \mathbb{R}^n\}.$$

Turns out f^* is always convex even when f is not, since it is the pointwise supremum of convex (in this case, affine) functions of y.

Under an additional technical assumption, we can indeed recover f as the conjugate of f^* .

Theorem 2.2. If F is convex and its epigraph is a closed set, then $f^{**} = f$.

We will not prove this theorem; see [BV, Exercise 3.39].

In fact f^{**} is the lower semi-continuous envelop of f, that is, the largest lower semi-continuous function upper-bounded by f. (We will not define semi-continuous here; just think of it as a weaker notion than continuity.)

Proposition 2.3 (Fenchel inequality). For any $x, y \in \mathbb{R}^n, \langle y, x \rangle \leq f^*(y) + f(x)$.

The proof follows from the definition of conjugate.

Examples of functions and their conjugates:

- Negative entropy. $f(x) = x \log x$, defined for $x \ge 0$. Then $f^*(y) = \sup_{x \ge 0} yx x \log x$ The supremum is achieved when $0 = \frac{d}{dx}(yx - x\log x) = y - x(\frac{1}{x}) - \log x \iff x = e^{y-1}$ Hence $f^*(y) = ye^{y-1} - e^{y-1}(y-1) = e^{y-1}$
- Strictly convex quadratic form. $f(x) = \frac{1}{2}x^{\top}Qx$, where Q is a symmetric positive definite matrix. Then $f^*(y) = \sup_x y^\top x - \frac{1}{2}x^\top Qx$.

The supremum is achieved when $0 = \nabla(y^\top x - \frac{1}{2}x^\top Qx) = y - Qx \iff x = Q^{-1}y$

Hence $f^*(y) = y^{\top}Q^{-1}y - \frac{1}{2}(y^{\top}Q^{-1}{}^{\top})Q(Q^{-1}y) = \frac{1}{2}y^{\top}Q^{-1}y$ • Log-sum-exp. $f(x) = \log(\sum_{1 \le i \le n} e^{x_i})$. [BV, Example 3.25] shows that $f^*(y) = \sum_i y_i \log y_i$, the negative entropy function, restricted to the probability simplex $(y \ge 0, \sum_{1 \le i \le n} y_i = 1)$.