Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) (25 points) Consider an undirected, unweighted graph $G = (V, E)$. Suppose $s, t \in V$ are two distinct vertices in G. For every vertex $a \in V$, consider starting the random walk at a, and let $p(a)$ denote the probability that the random walk reaches s before t. Write down the system of linear equations satisfied by the vector $p \in \mathbb{R}^V$. How is $p(a)$ related to a certain quantity in electric flow that we studied in class?

(2) Consider the barbell graph on $2n$ vertices. It is an undirected and unweighted graph, and is the disjoint union of two non-overlapping cliques, each containing n vertices, plus a single edge that has an endpoint in each clique. Altogether it has $2\binom{n}{2} + 1$ edges. (Lookup "barbell graph" on MathWorld for an illustration.)

    (a) (15 points) Show that the edge joining the two cliques has effective resistance 1, while any other edge has effective resistance $\Theta(1/n)$.

    (b) (10 points) Suppose we want to sparsify the barbell graph by randomly sampling m' edges to get a sparse graph H. Unlike the algorithm in class, we simply sample m' edges with replacement uniformly at random (and not proportional to their effective resistances). Show that whenever $m' = o(n^2)$, with probability approaching 1, the resulting graph H will not satisfy
$$\frac{1}{2}L_G \preccurlyeq L_H \preccurlyeq \frac{3}{2}L_G \ .$$

    This shows that sampling edges uniformly at random will fail to sparsify the graph. *Hint: What is the probability that the inter-clique edge appears in H?*

(3) (25 points) Let G be an undirected, unweighted, connected graph. Suppose the effective resistance of every edge is at most $1/k$ (by the effective resistance of an edge $e = (a, b)$, we mean the effective resistance $R_{\text{eff}}(a, b)$ of a and b). Prove that G is k-edge connected, that is, $E(S, \overline{S}) \geqslant k$ for every nonempty $S \subsetneq V$.

Hint: Prove the contrapositive: Suppose S is a cut such that $E(S, \overline{S}) = k$, show that any edge across this cut has effective resistance at least $1/k$.

(4) (25 points) Let G be an unweighted complete tri-partite graph. Its vertex set $V_1 \cup V_2 \cup V_3$ is the disjoint union of V_1, V_2 and V_3. Two vertices u and v are adjacent in G if and only if u and v belong to two different parts, so that $u \in V_i$ and $v \in V_j$ and $i \neq j$.

Prove that the normalized adjacency matrix of G has nonpositive second largest eigenvalue.