Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) (15 points) Suppose you are given a sequence of nonnegative real numbers $d_{i,j}$ $(1 \leqslant i < j \leqslant n)$, and you want to know whether there are points $v_1, \ldots, v_n$ in the n-dimensional Eucldean space such that their pairwise distances are exactly $d_{i,j}$ (that is, $\|v_i - v_j\|_2 = d_{i,j}$ for all $1 \leqslant i < j \leqslant n$).

Formulate this problem as the feasibility of a semidefinite program.

(2) (15 points) Let $\{f_\alpha\}_{\alpha \in I}$ be a collection of real-valued functions from $\mathbb{R}^n$, where I is an arbitrary index set. Suppose for every $\alpha \in I$, the function $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ is convex.

Define $f(x) \stackrel{\text{def}}{=} \sup_{\alpha \in I} f_\alpha(x)$ for every $x \in \mathbb{R}^n$. Prove that f is convex.

(3) (a) (10 points) An ellipsoid $\mathcal{E}_A$ centered at the origin is the image of the unit ball under the linear map A:

$$\mathcal{E}_A \stackrel{\text{def}}{=} \{Ax \mid x \in \mathbb{R}^n, \|x\| \leqslant 1\} ,$$

where A is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

If A is invertible, show that $\mathcal{E}_A$ is precisely the set of vectors $y \in \mathbb{R}^n$ such that $y^T(AA^T)^{-1}y \leqslant 1$.

(b) (20 points) If A and B are invertible linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, show that $\mathcal{E}_A \subseteq \mathcal{E}_B$ if and only if $AA^T \preccurlyeq BB^T$.

This shows that the semidefinite ordering $\preccurlyeq$ (known as Löewner ordering) corresponds to the inclusion relationship of certain ellipsoids. In fact part (b) still holds without assuming A and B to be invertible, but you do not need to consider that general case.

(4) (20 points) Given an n-by-n real symmetric matrix X, define

$$\exp X \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{X^k}{k!} .$$

Note that when $n = 1$, this definition reduces to the usual definition of the exponential function for real numbers.

Derive the Fenchel conjugate of

$$f(X) \stackrel{\text{def}}{=} \log(\operatorname{Tr} \exp X)$$

as a function on n-by-n real symmetric matrices. Here Tr denotes the trace function.

Useful Fact: $\nabla_X \operatorname{Tr} \exp X = \exp X$.

(5) (20 points) Let $\mathcal{S} \subseteq \mathcal{P}(\{1, \ldots, n\})$ be a family of subsets over a universe of size n (here $\mathcal{P}$ denotes the power set). The following program finds the maximum entropy probability distribution p

supported on $\mathcal{S}$ subject to marginal probability constraints:

$$\begin{aligned} \max \sum_{S\in\mathcal{S}} p_S \log \frac{1}{p_S} \\ \sum_{S\in\mathcal{S}} p_S = 1 \\ \text{for } 1 \leqslant i \leqslant n \qquad \sum_{S\in\mathcal{S},\, S\ni i} p_S = b_i \\ \forall S\in\mathcal{S} \quad p_S \geqslant 0 \end{aligned}$$

By considering the optimality condition of the Lagrangian, show that any maximizer satisfying $p_S > 0$ for all $S \in \mathcal{S}$ must be of the form

$$p_S = \frac{\prod_{i\in S} e^{\lambda_i}}{\sum_{T\in\mathcal{S}} \prod_{i\in T} e^{\lambda_i}}$$

for some real numbers λ_i.