
Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) Consider an undirected, unweighted graph $G = (V, E)$. Suppose $s, t \in V$ are two distinct vertices in G. For every vertex $a \in V$, consider starting the random walk at a, and let $p(a)$ denote the probability that the random walk reaches s before t.

Write down the system of linear equations satisfied by the vector $p \in \mathbb{R}^V$. How is $p(a)$ related to a certain quantity in electric flow that we studied in class?

(2) Consider the barbell graph on $2n$ vertices. It is an undirected and unweighted graph, and is the disjoint union of two non-overlapping cliques, each containing n vertices, plus a single edge that has an endpoint in each clique. Altogether it has $2\binom{n}{2} + 1$ edges. (Look up "barbell graph" on MathWorld for an illustration.)

(a) Show that the edge joining the two cliques has effective resistance 1, while any other edge has effective resistance $\Theta(1/n)$.

(b) Suppose we want to sparsify the barbell graph by randomly sampling m' edges to get a sparse graph H. Unlike the algorithm in class, we simply sample m' edges with replacement uniformly at random (and not proportional to their effective resistances). Show that whenever $m' = o(n^2)$, with probability approaching 1, the resulting graph H will not satisfy

$$\frac{1}{2} L_G \preccurlyeq L_H \preccurlyeq \frac{3}{2} L_G.$$

This shows that sampling edges uniformly at random will fail to sparsify the graph. *Hint: What is the probability that the inter-clique edge appears in H?*

(3) Let G be an undirected, unweighted, connected graph. Suppose the effective resistance of every edge is at most $1/k$ (by the effective resistance of an edge $e = (a, b)$, we mean the effective resistance of its endpoints a and b). Prove that G is k-edge connected, that is, $E(S, \overline{S}) \geqslant k$ for every nonempty $S \subsetneq V$.

Hint: Prove the contrapositive: Suppose S is a cut such that $E(S, \overline{S}) = k$, show that any edge across this cut has effective resistance at least $1/k$. The original definition of effective resistance will help you. The short proof is about 10 sentences long.

(4) Suppose $f : \{1, -1\}^n \to \{1, -1\}$ satifies

$$f(\mathbb{1}) f(x) f(y) f(xy) = 1 \qquad \text{for every } x, y \in \{1, -1\}^n, \tag{1}$$

where $xy \in \{1, -1\}^n$ denotes component-wise multiplication, and $\mathbb{1}$ denotes the all 1 vector. It is easy (but you do not need) to show that there exists some $S \subseteq [n]$ such that

$$\text{for every } T \subseteq [n], \qquad \hat{f}_T = \begin{cases} \pm 1 & \text{if } T = S \\ 0 & \text{otherwise} \end{cases}.$$

Here $\hat{f}_T = \mathbb{E}_{x\in\{1,-1\}^n}[f(x)\chi_T(x)]$ denotes the Fourier coefficient of f with respect to the character $\chi_T(x) = \prod_{i\in T} x_i$.

(a) Eq. (1) looks similar to the linearity condition discussed in class. How would you call a function satisfying Eq. (1)?

(b) Let $\mathcal{C}$ denotes the set of functions f satisfying Eq. (1), i.e. $\mathcal{C} = \{\pm\chi_T \mid T \subseteq [n]\}$. Consider the following test T to check whether $g \in \mathcal{C}$:

> Pick $x, y, z \in \{1,-1\}^n$ independently and uniformly at random, and accept if and only if
> $$g(x)g(y)g(z)g(xyz) = 1.$$

Show that T rejects with probability at least

$$\Delta(g, \mathcal{C}) \stackrel{\text{def}}{=} \min_{f\in\mathcal{C}} \Pr_{x\in\{1,-1\}^n}[g(x) \neq f(x)].$$

(5) Let P be the convex hull of $\{0, e_1, \ldots, e_n\}$ (i.e. the origin and the standard basis in $\mathbb{R}^n$). Let $\mathcal{E}$ be the minimum volume ellipsoid containing P.

(a) Recall that a full-dimensional ellipsoid is the image $\{Ax + c \mid x \in \mathbb{R}^n, \|x\| \leqslant 1\}$ of the unit ball under an invertible self-adjoint map $A : \mathbb{R}^n \to \mathbb{R}^n$ shifted by the center $c \in \mathbb{R}^n$. What are A and c for the minimum volume ellipsoid $\mathcal{E}$ of P? Explain why the ellipsoid described by your A and c has minimum volume among those containing P.

(b) Show that $c + \alpha(\mathcal{E} - c) \subseteq P$ implies $\alpha \leqslant 1/n$. In other words, $\mathcal{E}$ must be shrunk by a factor n to be contained inside P, and therefore factor $1/n$ is tight for the minimum volume ellipsoid to be contained inside an n-dimensional polytope.

You may find Sections 8.4.1 and 8.4.3 of Boyd–Vandenberghe useful. You may even answer parts (a) and (b) using another n-dimensional simplex if you wish.