# Pumping Lemma for Context-Free Languages

## CSCI 3130 Formal Languages and Automata Theory

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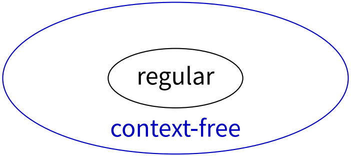



$$L_1 = \{\mathsf{a}^n\mathsf{b}^n \mid n \geqslant 0\}$$
$$L_2 = \{z \mid z \text{ has the same number of a's and b's}\}$$
$$L_3 = \{\mathsf{a}^n\mathsf{b}^n\mathsf{c}^n \mid n \geqslant 0\}$$
$$L_4 = \{zz^R \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$$
$$L_5 = \{zz \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$$

These languages are not regular

Are they context-free?

# An attempt

$$L_3 = \{\mathsf{a}^n\mathsf{b}^n\mathsf{c}^n \mid n \geqslant 0\}$$

Let's try to design a CFG or PDA

$S \to \mathsf{a}B\mathsf{c} \mid \varepsilon$

$B \to$ ???

read a / push x

read b / pop x

???

Suppose we could construct some CFG $G$ for $L_3$

e.g.

$S \to CC \mid BC \mid \text{a}$

$B \to CS \mid \text{b}$

$C \to SB \mid \text{c}$

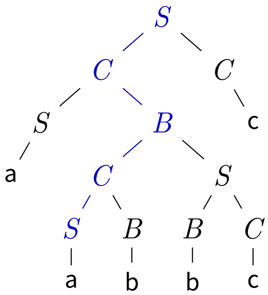
How does a long derivation look like?

$$
\begin{aligned}
S &\Rightarrow CC \\
&\Rightarrow SBC \\
&\Rightarrow SCSC \\
&\Rightarrow SSBSC \\
&\Rightarrow SSBBCC \\
&\Rightarrow \text{a}SBBCC \\
&\Rightarrow \text{aa}BBCC \\
&\Rightarrow \text{aab}BCC \\
&\Rightarrow \text{aabb}CC \\
&\Rightarrow \text{aabbc}C \\
&\Rightarrow \text{aabbcc}
\end{aligned}
$$

# Repetition in long derivations

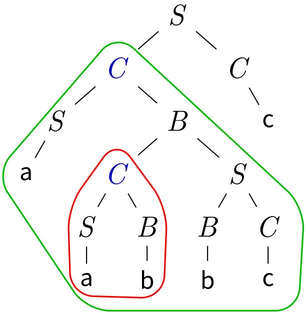
If a derivation is long enough, some variable must appear twice on the same root-to-leave path in a parse tree

$$
\begin{aligned}
S &\Rightarrow CC \\
&\Rightarrow SBC \\
&\Rightarrow SCSC \\
&\Rightarrow SSBSC \\
&\Rightarrow SSBBCC \\
&\Rightarrow \mathsf{a}SBBCC \\
&\Rightarrow \mathsf{aa}BBCC \\
&\Rightarrow \mathsf{aab}BCC \\
&\Rightarrow \mathsf{aabb}CC \\
&\Rightarrow \mathsf{aabbc}C \\
&\Rightarrow \mathsf{aabbcc}
\end{aligned}
$$

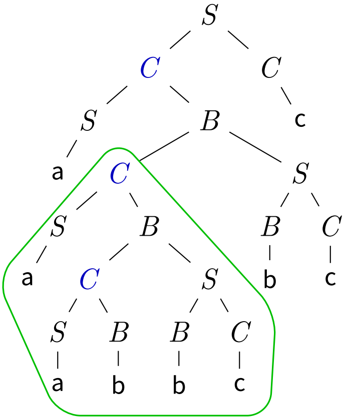


# Pumping example

Then we can "cut and paste" part of parse tree




aabbcc




aaabbcbcc

# Pumping example

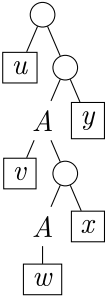
We can repeat this many times

aabbcc $\Rightarrow$ aaabbcbcc $\Rightarrow$ aaaabbcbcbcc $\Rightarrow \ldots$

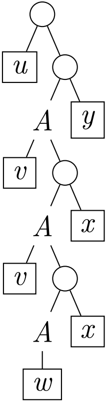
$\Rightarrow \mathsf{a}(\mathsf{a})^i\mathsf{b}(\mathsf{bc})^i\mathsf{c}$

Every sufficiently large derivation will have a middle part that can be repeated indefinitely

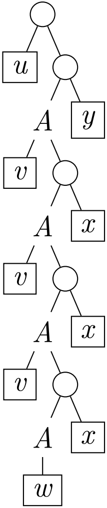
# Pumping in general



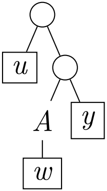

$uvwxy$




$uv^2wx^2y$




$uv^3wx^3y$




$uwy$

# Example

$$L_3 = \{\mathsf{a}^n\mathsf{b}^n\mathsf{c}^n \mid n \geqslant 0\}$$

If $L_3$ has a context-free grammar $G$, then for any sufficiently long $s \in L(G)$

$s$ can be split into $s = uvwxy$ such that $L(G)$ also contains $uv^2wx^2y$, $uv^3wx^3y, \ldots$

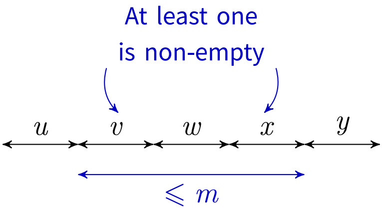
What happens if $s = \mathsf{a}^m\mathsf{b}^m\mathsf{c}^m$

No matter how it is split, $uv^2wx^2y \notin L_3$

# Pumping lemma for context-free languages

For every context-free language $L$
There exists a number $m$ such that for every long string $s$ in $L$ ($|s| \geqslant m$),
we can write $s = uvwxy$ where

1. $|vwx| \leqslant m$
2. $|vx| \geqslant 1$
3. For every $i \geqslant 0$, the string $uv^iwx^iy$ is in $L$



# Pumping lemma for context-free languages

To prove $L$ is not context-free, it is enough to show that

For every $m$ there is a long string $s \in L$, $|s| \geqslant m$, such that for every way of writing $s = uvwxy$ where

1. $|vwx| \leqslant m$
2. $|vx| \geqslant 1$

there is $i \geqslant 0$ such that $uv^iwx^iy$ is not in $L$

# Using the pumping lemma

$$L_3 = \{\mathsf{a}^n\mathsf{b}^n\mathsf{c}^n \mid n \geqslant 0\}$$

1. for every $m$
2. there is $s = \mathsf{a}^m\mathsf{b}^m\mathsf{c}^m$ (longer than $m$ symbols)
3. no matter how the pumping lemma splits $s$ into $uvwxy$ ($|vwx| \leqslant m, |vx| \geqslant 1$)
4. $uv^2wx^2y \notin L_3$ (but why?)

# Using the pumping lemma

Case 1: $v$ or $x$ contains two kinds of symbols

$$\text{aa}\underbrace{\text{aabb}}_{v}\text{bbcccc}$$

Then $uv^2wx^2y \notin L_3$ because the pattern is wrong

Case 2: $v$ and $x$ both contain one kind of symbol

$$\text{a}\underbrace{\text{aaa}}_{v}\text{b}\underbrace{\text{bb}}_{x}\text{bbcccc}$$

Then $uv^2wx^2y$ does not have the same number of a's, b's and c's

Conclusion: $uv^2wx^2y \notin L_3$

Which is context-free?

$L_1 = \{\mathsf{a}^n \mathsf{b}^n \mid n \geqslant 0\}$ ✓

$L_2 = \{z \mid z \text{ has the same number of a's and b's}\}$ ✓

$L_3 = \{\mathsf{a}^n \mathsf{b}^n \mathsf{c}^n \mid n \geqslant 0\}$ ✗

$L_4 = \{zz^R \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$ ✓

$L_5 = \{zz \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$

# Example

$$L_5 = \{zz \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$$

1. for every $m$
2. there is $s = \mathsf{a}^m\mathsf{b}\mathsf{a}^m\mathsf{b}$ (longer than $m$ symbols)
3. no matter how the pumping lemma splits $s$ into $uvwxy$ ($|vwx| \leqslant m, |vx| \geqslant 1$)
4. Is $uv^2wx^2y \notin L_5$?

# Example

$$L_5 = \{zz \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$$

1. for every $m$
2. there is $s = \mathsf{a}^m\mathsf{b}\mathsf{a}^m\mathsf{b}$ (longer than $m$ symbols)
3. no matter how the pumping lemma splits $s$ into $uvwxy$ ($|vwx| \leqslant m, |vx| \geqslant 1$)
4. Is $uv^2wx^2y \notin L_5$?

$$\mathsf{aaa}\underbrace{\mathsf{a}}_{v}\mathsf{abaa}\underbrace{\mathsf{a}}_{x}\mathsf{aab}$$

# Example

$$L_5 = \{zz \mid z \in \{\mathsf{a}, \mathsf{b}\}^*\}$$

1. for every $m$
2. there is $s = \mathsf{a}^m\mathsf{b}^m\mathsf{a}^m\mathsf{b}^m$ (longer than $m$ symbols)
3. no matter how the pumping lemma splits $s$ into $uvwxy$ ($|vwx| \leqslant m, |vx| \geqslant 1$)
4. Is $uv^iwx^iy \notin L_5$ for some $i$?

Recall that $|vwx| \leqslant m$

# Example

Three cases

Case 1 $\quad$ aaa $\underbrace{\text{aabbb}}_{vwx}$ bbaaaaabbbbb

$vwx$ is in the first half of $a^m b^m a^m b^m$

Case 2 $\quad$ aaaaabb $\underbrace{\text{bbbaa}}_{vwx}$ aaabbbbb

$vwx$ is in the middle part of $a^m b^m a^m b^m$

Case 3 $\quad$ aaaaabbbbbaaa $\underbrace{\text{aabbb}}_{vwx}$ bb

$vwx$ is in the second half of $a^m b^m a^m b^m$

# Example

Apply pumping lemma with $i = 0$

Case 1 aaa $\underbrace{\text{aabbb}}_{vwx}$ bbaaaaabbbbb

$uwy$ becomes $\mathsf{a}^j\mathsf{b}^k\mathsf{a}^m\mathsf{b}^m$, where $j < m$ or $k < m$

Case 2 aaaaabb $\underbrace{\text{bbbaa}}_{vwx}$ aaabbbbb

$uwy$ becomes $\mathsf{a}^m\mathsf{b}^j\mathsf{a}^k\mathsf{b}^m$, where $j < m$ or $k < m$

Case 3 aaaaabbbbbaaa $\underbrace{\text{aabbb}}_{vwx}$ bb

$uwy$ becomes $\mathsf{a}^m\mathsf{b}^m\mathsf{a}^j\mathsf{b}^k$, where $j < m$ or $k < m$

Not of the form $zz$

This covers all cases, so $L_5$ is not context-free