Irregular Languages

CSCI 3130 Formal Languages and Automata Theory

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Non-regular languages

Are there irregular languages?

Candidate from last lecture:

$$L = \{0^n 1 0^n 1 \mid n \geqslant 0\}$$

(duplicate of language of $0^*1 = \{1, 01, 001, 0001, \ldots\}$)

Non-regular languages

Are there irregular languages?

Candidate from last lecture:

$$L = \{0^n 1 0^n 1 \mid n \geqslant 0\}$$

(duplicate of language of $0^* 1 = \{1, 01, 001, 0001, \ldots\}$)

Why do we believe it is irregular?

Seems to require a "DFA" with infinitely many states

After reading the first half, need to remember number of zeros so far

11, 0101, 001001, 00010001, …

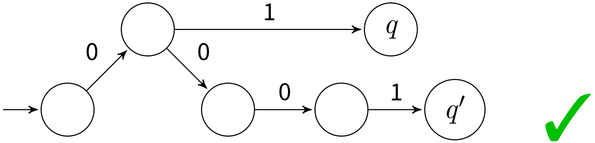
Infinitely many possibilities

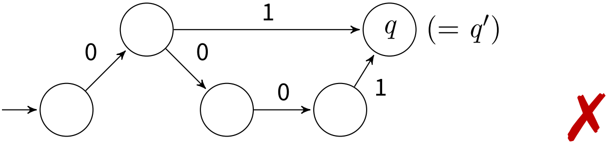
Let's formally prove this intuition

Distinct states for 01 and 0001

Claim

If a deterministic automaton accepts $L = \{0^n 1 0^n 1 \mid n \geqslant 0\}$, the state q it reaches upon reading 01 must be different from the state q' it reaches upon reading 0001





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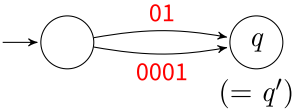
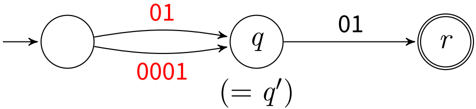


✗

Why not?

Reason: after going to q, if it reads 01 and reaches r …

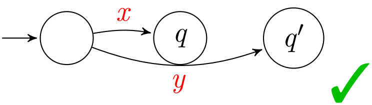


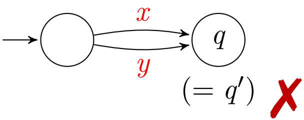
If r is not accepting, it rejects 0101 ✗

If r is accepting state, it accepts 000101 ✗

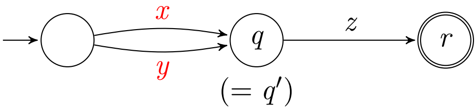
General case: distinguishable strings

If a deterministic automaton accepts L, if there are strings x and y such that $xz \in L$ but $yz \notin L$, then the automaton must be in two different states upon reading x and y





Reason:



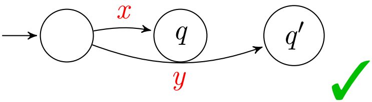
If r is not accepting, it rejects xz ✗

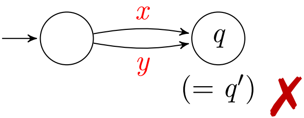
If r is accepting state, it accepts yz ✗

Distinguishable strings

x and y are distinguishable by L if for some string z,
we have $xz \in L$ and $yz \notin L$ (or the other way round)

If x and y are distinguishable by L, any deterministic automaton accepting L must reach different states upon reading x and y

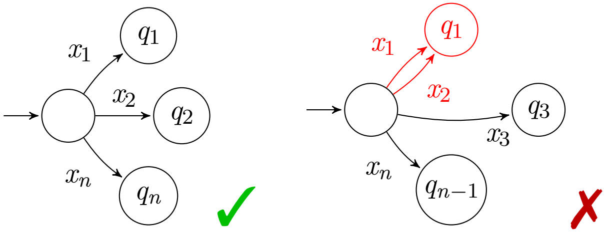




Requires many states

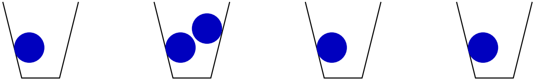
Strings $x_1, \ldots, x_n$ are called pairwise distinguishable by L if every pair x_i and x_j are distinguishable by L, for any $i \neq j$.

If strings $x_1, \ldots, x_n$ are pairwise distinguishable by L, any deterministic automaton accepting L must have at least n states



Pigeonhole principle

If you put 5 balls into 4 bins,
then (at least) two balls end up in the same bin



More generally

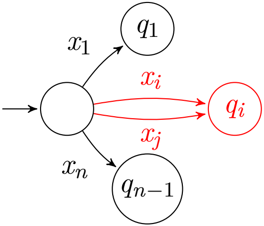
If you put n balls into (at most) $n-1$ bins,
then (at least) two balls end in the same bin

Pigeonhole principle

Requires many states

If strings $x_1, \ldots, x_n$ are pairwise distinguishable by L, any deterministic automaton accepting L must have at least n states

Otherwise:



If there are (at most) $n-1$ states, by pigeonhole principle, two different strings x_i and x_j must end up at the same state, but:

If x_i and x_j are distinguishable by L, any deterministic automaton accepting L must reach different states upon reading x_i and x_j ✗

0^n10^n1 is not regular

Suffices find an infinitely sequence of strings that are pairwise distinguishable by $L = \{0^n10^n1 \mid n \geqslant 0\}$

After reading the first half, need to remember number of zeros so far

11, 0101, 001001, 00010001, …

1, 01, 001, 0001, . . . are pairwise distinguishable by L

Why are 0^i1 and 0^j1 distinguishable by L? $(i \neq j)$

0^n10^n1 is not regular

Suffices find an infinitely sequence of strings that are pairwise distinguishable by $L = \{0^n10^n1 \mid n \geqslant 0\}$

After reading the first half, need to remember number of zeros so far

11, 0101, 001001, 00010001, …

1, 01, 001, 0001, . . . are pairwise distinguishable by L

Why are 0^i1 and 0^j1 distinguishable by L? $(i \neq j)$

Take $z = 0^i1$

$0^i10^i1 \in L$ $\quad$ $0^j10^i1 \notin L$

Which of these are (ir)regular?

$$L_1 = \{x \mid x \text{ has the same number of 0s and 1s}\}$$
$$L_2 = \{\mathtt{0}^n\mathtt{1}^m \mid n > m \geqslant 0\}$$
$$L_3 = \{x \mid x \text{ has the same number of patterns 01 and 11}\}$$
$$L_4 = \{x \mid x \text{ has the same number of patterns 01 and 10}\}$$
$$L_5 = \{x \mid x \text{ has a different number of 0s and 1s}\}$$

$L_1 =$ Same number of 0s and 1s

Why does it require infinitely many states to accept?

L_1 = Same number of 0s and 1s

Why does it require infinitely many states to accept?

Need to remember number of 0s (or 1s) read so far

ε, 0, 00, 000, ... are pairwise distinguishable by L_1

Why are 0^i and 0^j distinguishable by L_1? $(i \neq j)$

$L_1 =$ Same number of 0s and 1s

Why does it require infinitely many states to accept?

Need to remember number of 0s (or 1s) read so far

ε, 0, 00, 000, $\ldots$ are pairwise distinguishable by L_1

Why are 0^i and 0^j distinguishable by L_1? $\quad (i \neq j)$

Take $z = 1^i$

$0^i 1^i \in L_1 \qquad 0^j 1^i \notin L_1$

$L_2 = \{0^n 1^m \mid n > m\}$

Like L_1, need to remember number of 0s read so far

ε, 0, 00, 000, $\ldots$ are pairwise distinguishable by L_2

Why are 0^i and 0^j distinguishable by L_2? $\quad (i > j)$

$L_2 = \{0^n 1^m \mid n > m\}$

Like L_1, need to remember number of 0s read so far

$\varepsilon, 0, 00, 000, \ldots$ are pairwise distinguishable by L_2

Why are 0^i and 0^j distinguishable by L_2? $(i > j)$

Take $z = 1^{i-1}$

$0^i 1^{i-1} \in L_2 \qquad 0^j 1^{i-1} \notin L_2$

$L_3 =$ same number of 01s and 11s

Need to remember the number of 01s read so far

ε, 01, 0101, 010101, $\ldots$ are pairwise distinguishable by L_3

Why are $(01)^i$ and $(01)^j$ distinguishable by L_3? $\quad (i > j)$

$L_3 =$ same number of 01s and 11s

Need to remember the number of 01s read so far

ε, 01, 0101, 010101, $\ldots$ are pairwise distinguishable by L_3

Why are $(01)^i$ and $(01)^j$ distinguishable by L_3? $(i > j)$

Take $z = 1^i$

$(01)^i 1^i \in L_3$ $(01)^j 1^i \notin L_3$

Example: 010101111 $(i = 3)$

L_4 = same number of 01s and 10s

ε, 01, 0101, 010101, . . . are pairwise distinguishable by L_4

Why are $(01)^i$ and $(01)^j$ distinguishable by L_4? $(i > j)$

Take $z = (10)^i$

$(01)^i(10)^i \in L_4$ $\quad$ $(10)^j(10)^i \notin L_4$

Example: 010101101010 $(i = 3)$

$L_4 =$ same number of 01s and 10s

~~ε, 01, 0101, 010101, $\ldots$ are pairwise distinguishable by L_4~~

~~Why are $(01)^i$ and $(01)^j$ distinguishable by L_4? $(i > j)$~~

~~Take $z = (10)^i$~~

~~$(01)^i(10)^i \in L_4$ $\quad$ $(10)^j(10)^i \notin L_4$~~

Example: 010101101010 $\quad (i = 3)$

In fact, $(01)^j(10)^i \in L_4$ because there are as many 01 as 10

In fact, L_4 is regular (see Week 2 tutorial)

L_5 = different number of 0s and 1s

Is L_5 irregular?

$L_5 =$ different number of 0s and 1s

Is L_5 irregular?

Yes

If L_5 were regular, then so is

$$\overline{L_5} = L_1 = \{x \mid x \text{ has the same number of 0s and 1s}\}$$

But we saw that L_1 is irregular, therefore so is L_5

An exercise

$L_6 =$ properly nested strings of parentheses $\qquad \Sigma = \{(,)\}$

(), (()), ()() are in L_6
(,),)(are not

Exercise: show that L_6 is irregular
What does it mean?

An exercise

$L_6 =$ properly nested strings of parentheses $\qquad \Sigma = \{(,)\}$

(), (()), ()() are in L_6
(,),)(are not

Exercise: show that L_6 is irregular
What does it mean?

Language $=$ computational problem
DFA $=$ machine with finite memory

L_6 is irregular $\Rightarrow$ checking whether (arbitrarily long) strings are properly nested requires unbounded amount of memory