Undecidable Problems for CFGs

CSCI 3130 Formal Languages and Automata Theory

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Decidable vs undecidable

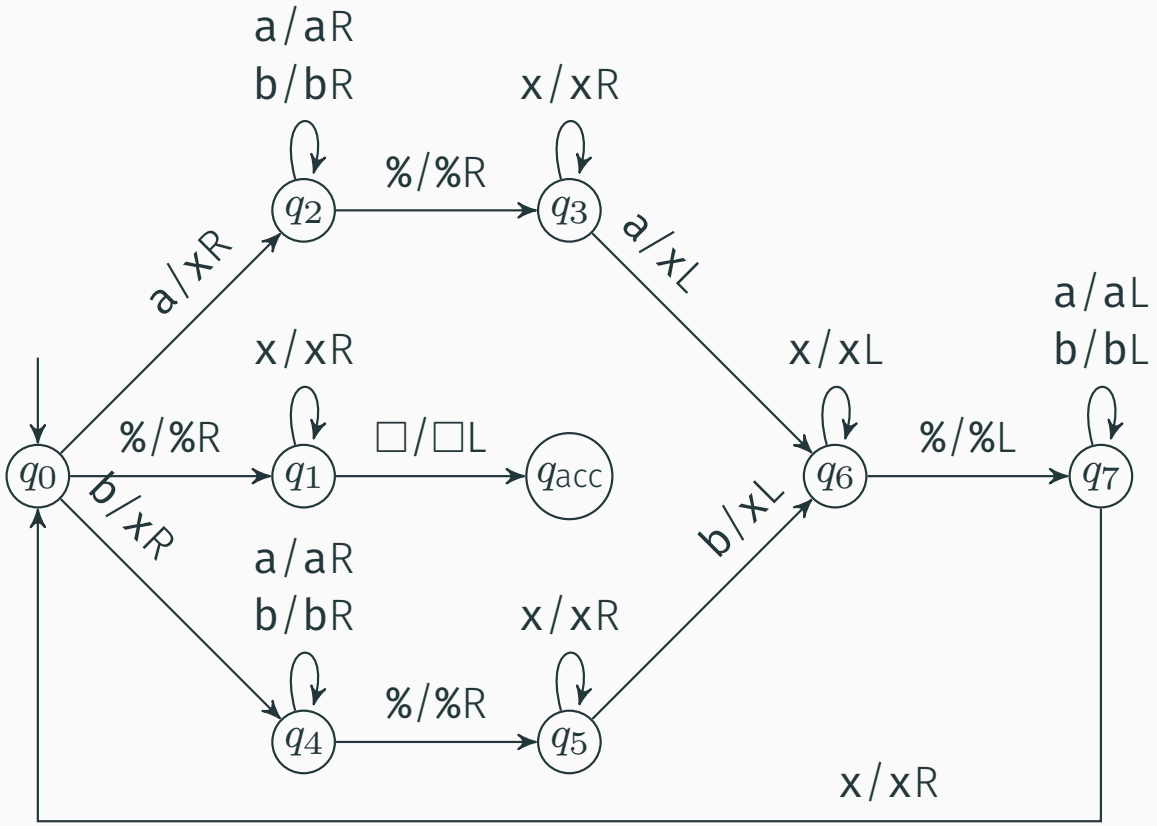
Decidable	Undecidable
DFA D accepts w	TM M accepts w
CFG G generates w	TM M halts on w
DFAs D and D' accept the same inputs	TM M accepts some input
	TM M and M' accept the same inputs

CFG G generates all inputs?

CFG G is ambiguous?

Representing computation

$$L_1 = \{w\%w \mid w \in \{\mathsf{a}, \mathsf{b}\}^*\}$$



q_0 a̲bb%abb

q_2 x̲bb%abb

⋮

q_2 xbb%̲abb

q_3 xbb%a̲bb

q_6 xbb%̲xbb

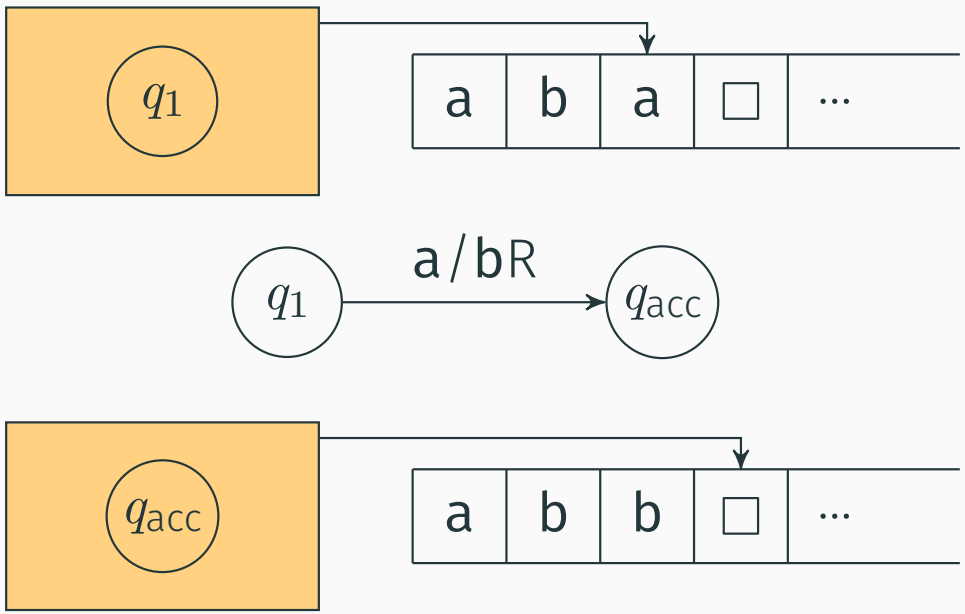
⋮

q_1 xxx%xxx□̲

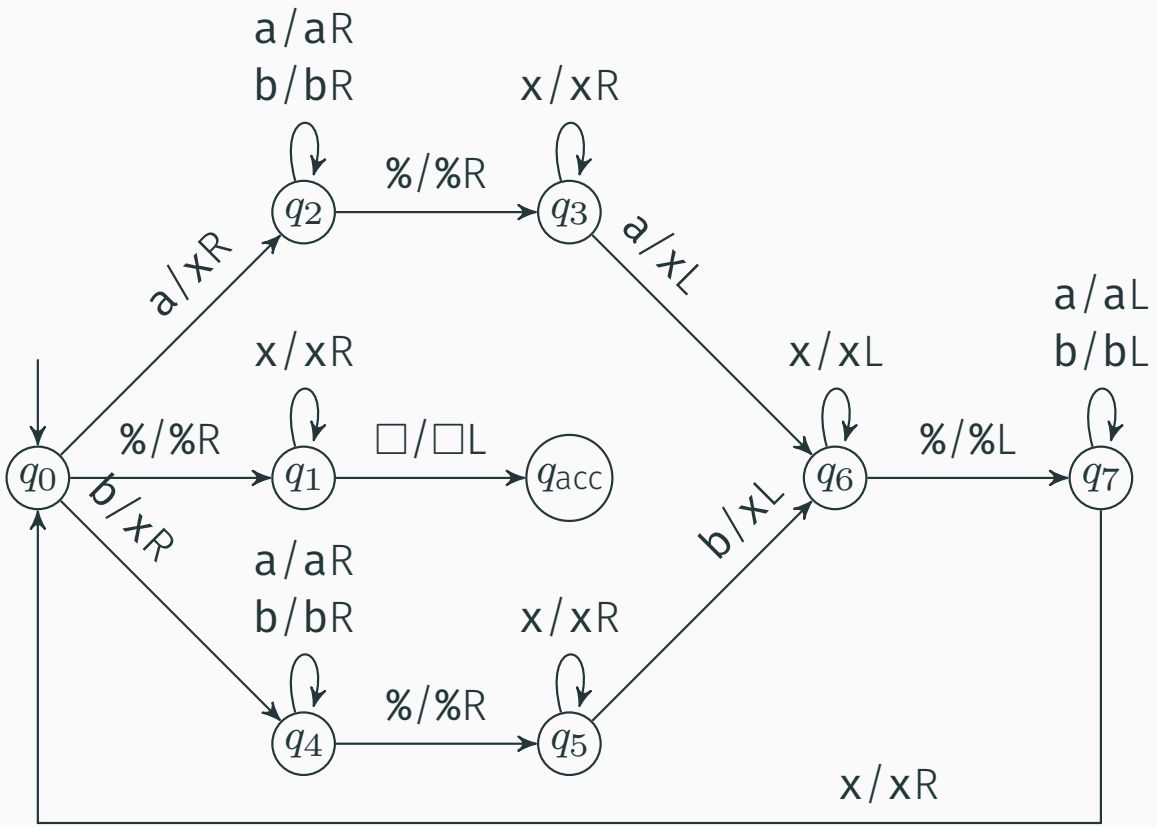
q_{acc} xxx%xxx̲

Configurations

A configuration consists of current state, head position, and tape contents



Computation history



q_0 abb%abb
x q_2 bb%abb
⋮
xbb q_2 %abb
xbb% q_3 abb
xbb q_2 %xbb
⋮
xxx%xxx q_1
xxx%xx q_{acc} x

computation history

Computation histories as strings

If M halts on w, the computation history of (M, w) is the sequence of configurations $C_1, \ldots, C_k$ that M goes through on input w

q_0 ab%ab
x q_2 b%ab
⋮
xx%xx q_1
xx%x q_{acc} x

$$\#\underbrace{q_0\text{ab\%ab}}_{C_1}\#\underbrace{\text{x}q_1\text{b\%ab}}_{C_2}\#\ldots\#\underbrace{\text{xx\%x}q_{\text{acc}}\text{x}}_{C_k}\#$$

The computation history can be written as a string h over alphabet $\Gamma \cup Q \cup \{\#\}$

accepting history: M accepts $w \Leftrightarrow q_{\text{acc}}$ appears in h

rejecting history: M rejects $w \Leftrightarrow q_{\text{rej}}$ appears in h

Undecidable problems for CFGs

$$\text{ALL}_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG that generates all strings}\}$$

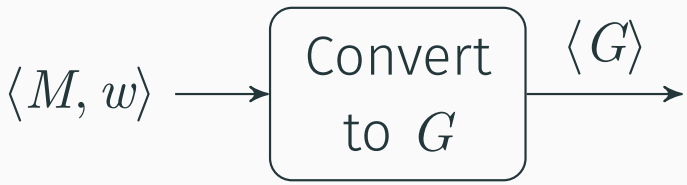
The language ALL_{CFG} is undecidable

We will argue that

If ALL_{CFG} can be decided, so can $\overline{A_{\text{TM}}}$

$$\overline{A_{\text{TM}}} = \{\langle M, w \rangle \mid M \text{ is a TM that rejects or loops on } w\}$$

Undecidable problems for CFGs



G fails to generate some string

$\Updownarrow$

M accepts w

G generates all strings if M rejects or loops on w

G fails to generate some string if M accepts w

The alphabet of G will be $\Gamma \cup Q \cup \{\#\}$

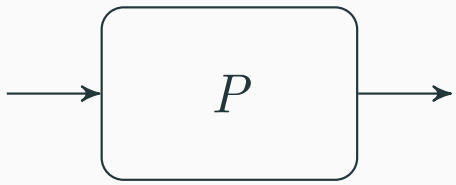
G will generate all strings except

accepting computation history of (M, w)

First we construct a PDA P, then convert it to CFG G

Undecidablility via computation histories

candidate computation history h of (M, w)



accept everything except accepting h

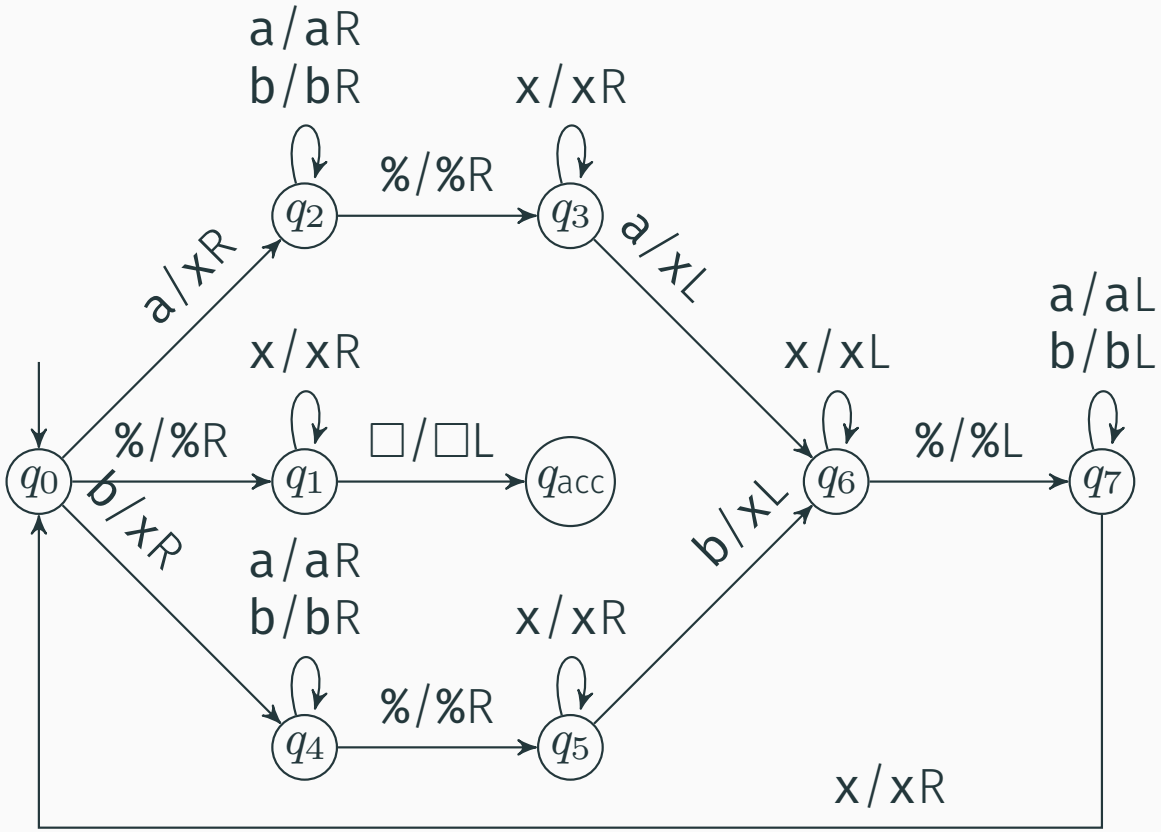
$\#q_0$ab%ab#xq_1b%ab#...#xx%xq_{acc}x# $\Rightarrow$ Reject

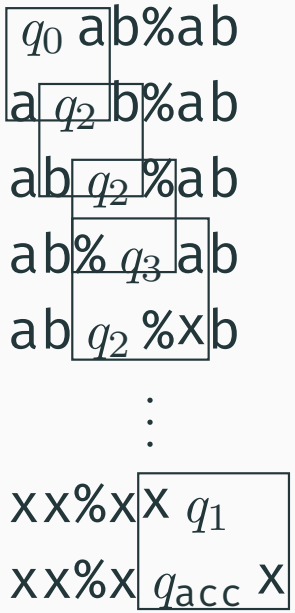
$P =$ on input h (try to spot a mistake in h)

- If h is not of the form $\#w_1\#w_2\#...\#w_k\#$, accept
- If $w_1 \neq q_0 w$ or w_k does not contain q_{acc}, accept
- If two consecutive blocks $w_i\#w_{i+1}$ do not follow from the transitions of M, accept

Otherwise, h must be an accepting history, reject

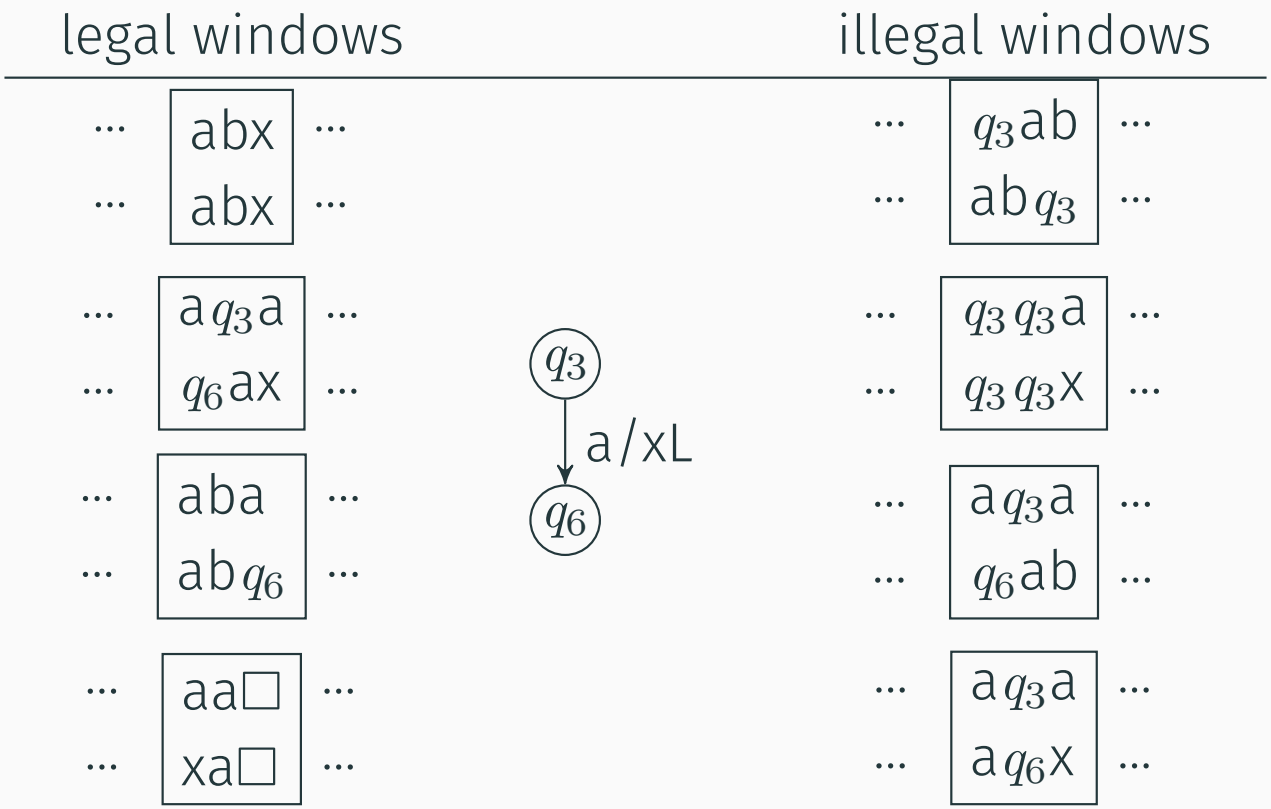
Computation is local





Changes between configurations always occur around the head

Legal and illegal transitions windows



Implementing P

If two consecutive blocks $w_i\#w_{i+1}$ do not follow from the transitions of M, accept

#xb%$q_3$ab
#xbq_5%xb

For every position of w_i:

Remember offset from # in w_i on stack

Remember first row of window in state

After reaching the next #:

Pop offset from # from stack as you consume input

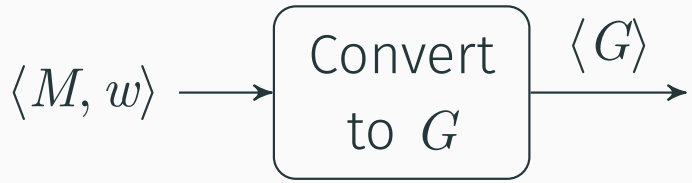
Remember second row of window in state

If window is illegal, accept; Otherwise reject

The computation history method

$$\text{ALL}_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG that generates all strings}\}$$

If ALL_{CFG} can be decided, so can $\overline{A_{\text{TM}}}$



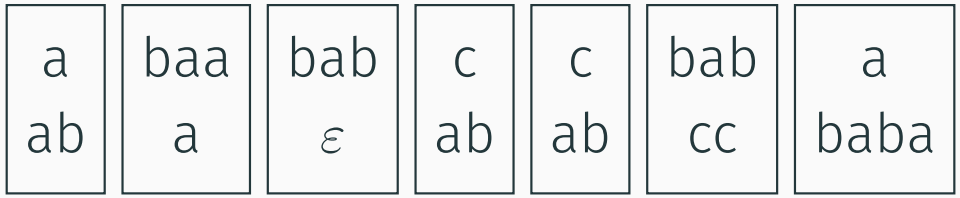
G accepts all strings except accepting computation history of (M, w)

We first construct a PDA P, then convert it to CFG G

Post Correspondence Problem

Input: A finite set of tiles, each containing a pair of strings

Given an infinite supply of tiles from a particular set, can you match top and bottom?



Top and bottom are both abaababccbaba

Undecidability of PCP

$$\text{PCP} = \{\langle C \rangle \mid$$
C is a collection of tiles that contains a top-bottom match$\}$

Next lecture we will show (using computation history method)

The language PCP is undecidable

Ambiguity of CFGs

$$\text{AMB} = \{\langle G \rangle \mid G \text{ is an ambiguous CFG}\}$$

The language AMB is undecidable

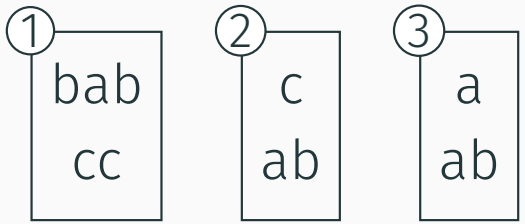
We will argue that

If AMB can be decided, then so can PCP

C (collection of tiles) $\longmapsto$ G (CFG)

If C can be matched, then G is ambiguous
If C cannot be matched, then G is unambiguous

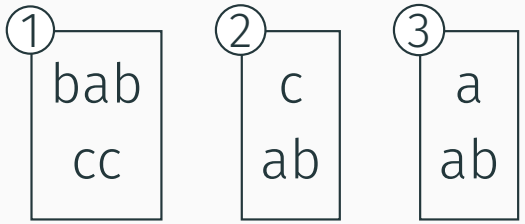
First, let's number the tiles



Encoding the tiles in a grammar

C (collection of tiles) $\longmapsto$ G (CFG)

Example:



Terminals: a, b, c, 1, 2, 3 Variables: S, T, B

Productions:

$S \to T \mid B$

$T \to \text{bab}\,T1$
$B \to \text{cc}\,B1$
$T \to \text{bab1}$
$B \to \text{cc1}$

$T \to \text{c}\,T2$
$B \to \text{ab}\,B2$
$T \to \text{c2}$
$B \to \text{ab2}$

$T \to \text{a}\,T3$
$B \to \text{ab}\,B3$
$T \to \text{a3}$
$B \to \text{ab3}$

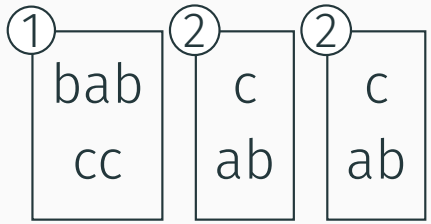
In general:

yields

$T \to \alpha\,Tn$ $T \to \alpha n$
$B \to \beta\,Bn$ $B \to \beta n$

Matching sequence implies ambiguous grammar

Each sequence of tiles gives a pair of derivations



$$S \Rightarrow T \Rightarrow \text{bab}\,T1 \Rightarrow \text{babc}\,T21 \Rightarrow \text{babcc}221$$

$$S \Rightarrow B \Rightarrow \text{cc}\,B1 \Rightarrow \text{ccab}\,B21 \Rightarrow \text{ccabab}221$$

If the tiles match, these two derive the same string

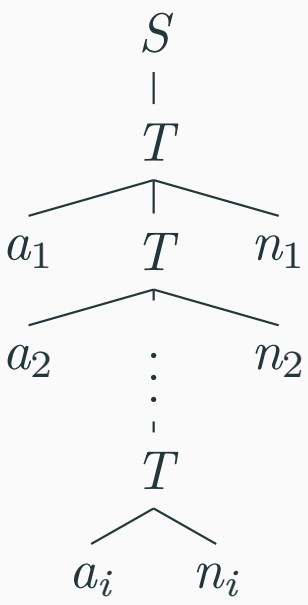
(with different parse trees)

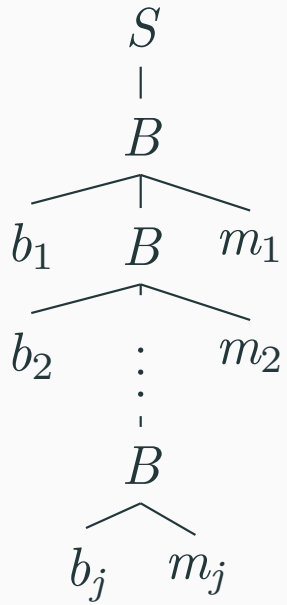
Ambiguous grammar implies matching sequence

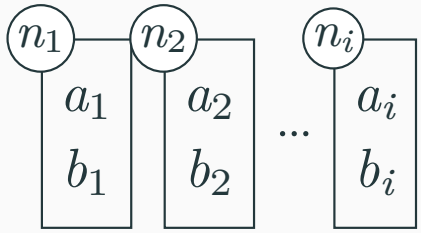
C (collection of tiles) $\longmapsto$ G (CFG)

If C can be matched, then G is ambiguous ✓
If C cannot be matched, then G is unambiguous ✓

If G is ambiguous, then the two parse trees look like







Therefore $n_1 n_2 \dots n_i = m_1 m_2 \dots m_j$, and there is a match