Decidability

CSCI 3130 Formal Languages and Automata Theory

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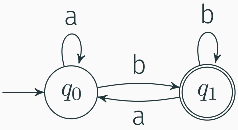
Chinese University of Hong Kong

Problems about automata

Does



accept input abb?

We can formulate this question as a language

$$A_{\text{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Is A_{DFA} decidable?

One possible way to encode a DFA $D = (Q, \Sigma, \delta, q_0, F)$ and input w

$$\texttt{(}\underbrace{\texttt{(q0,q1)}}_{Q}\underbrace{\texttt{(a,b)}}_{\Sigma}\underbrace{\texttt{((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))}}_{\delta}\underbrace{\texttt{(q0)}}_{q_0}\underbrace{\texttt{(q1)}}_{F}\texttt{)(}\underbrace{\texttt{abb}}_{w}\texttt{)}$$

Problems about automata

$A_{\text{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$

Pseudocode:

On input $\langle D, w\rangle$, where $D = (Q, \Sigma, \delta, q_0, F)$

Set $q \leftarrow q_0$
For $i \leftarrow 1$ to length(w)
    $q \leftarrow \delta(q, w_i)$
If $q \in F$ accept, else reject

TM description:

On input $\langle D, w\rangle$, where D is a DFA, w is a string

Simulate D on input w
If simulation ends in an accept state, accept; else reject

Problems about automata

$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$

Turing machine details:

Check input is in correct format

(Transition function is complete, no duplicate transitions)

Perform simulation:

$\texttt{((}\dot{\texttt{q}}\texttt{0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(}\dot{\texttt{a}}\texttt{bb)}$

$\texttt{((}\dot{\texttt{q}}\texttt{0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(a}\dot{\texttt{b}}\texttt{b)}$

$\texttt{((q0,}\dot{\texttt{q}}\texttt{1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(ab}\dot{\texttt{b}}\texttt{)}$

$\texttt{((q0,}\dot{\texttt{q}}\texttt{1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(abb}\dot{\texttt{)}}$

Problems about automata

$A_{\text{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$

Turing machine details:

Check input is in correct format

(Transition function is complete, no duplicate transitions)

Perform simulation: (very high-level)

Put markers on start state of D and first symbol of w

Until marker for w reaches last symbol:

Update both markers

If state marker is on accepting state, accept; else reject

Conclusion: A_{DFA} is decidable

Acceptance problems about automata

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\} \quad ✓$$

$$A_{\mathsf{NFA}} = \{\langle N, w\rangle \mid N \text{ is an NFA that accepts input } w\}$$

$$A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$$

Which of these is decidable?

Acceptance problems about automata

$$A_{\text{NFA}} = \{\langle N, w\rangle \mid N \text{ is an NFA that accepts input } w\}$$

The following TM decides A_{NFA}:

On input $\langle N, w\rangle$ where N is an NFA and w is a string

Convert N to a DFA D using the conversion procedure from Lecture 3
Run TM M for A_{DFA} on input $\langle D, w\rangle$
If M accepts, accept; else reject

Conclusion: A_{NFA} is decidable ✓

Acceptance problems about automata

$A_{\text{REX}} = \{\langle R, w\rangle \mid R$ is a regular expression that generates $w\}$

The following TM decides A_{REX}

On input $\langle R, w\rangle$, where R is a regular expression and w is a string

Convert R to NFA N using the conversion procedure from Lecture 4
Run the TM M' for A_{NFA} on input $\langle N, w\rangle$
If M' accepts, accept; else reject

Conclusion: A_{REX} is decidable ✓

Other problems about automata

$$\mathsf{MIN}_{\mathsf{DFA}} = \{\langle D\rangle \mid D \text{ is a minimal DFA}\}$$

$$\mathsf{EQ}_{\mathsf{DFA}} = \{\langle D_1, D_2\rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$$

$$E_{\mathsf{DFA}} = \{\langle D\rangle \mid D \text{ is a DFA and } L(D) \text{ is empty}\}$$

Which of the above is decidable?

Other problems about automata

$$\text{MIN}_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a minimal DFA}\}$$

The following TM decides MIN_{DFA}

On input $\langle D \rangle$, where D is a DFA

Run the DFA minimization algorithm from Lecture 7

If every pair of states is distinguishable, accept; else reject

Conclusion: MIN_{DFA} is decidable ✓

Other problems about automata

$\text{EQ}_{\text{DFA}} = \{\langle D_1, D_2 \rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$

The following Turing machine S decides EQ_{DFA}

TM S: On input $\langle D_1, D_2 \rangle$, where D_1 and D_2 are DFAs

Run DFA minimization algorithm on D_1 to obtain a minimal DFA D'_1
Run DFA minimization algorithm on D_2 to obtain a minimal DFA D'_2
If $D'_1 = D'_2$, accept; else reject

Conclusion: EQ_{DFA} is decidable ✓

Other problems about automata

$E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) \text{ is empty}\}$

The following TM T decides E_{DFA}

Turing machine M: On input $\langle D \rangle$, where D is a DFA

Run the TM S for EQ_{DFA} on input $\langle D, D' \rangle$,

where D' is any DFA that accepts no input, such as 

If S accepts, accept; else reject

Conclusion: E_{DFA} is decidable ✓

Problems about context-free grammars

$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$

L where L is a context-free language

$\mathsf{EQ}_{\mathsf{CFG}} = \{\langle G_1, G_2\rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$

Which of the above is decidable?

Problems about context-free grammars

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$$

The following TM V decides A_{CFG}

TM V: On input $\langle G, w\rangle$, where G is a CFG and w is a string

Eliminate the ε- and unit productions from G
Convert G into Chomsky Normal Form G'
Run Cocke–Younger–Kasami algorithm on $\langle G', w\rangle$
If the CYK algorithm finds a parse tree, accept; else reject

Conclusion: A_{CFG} is decidable ✓

Problems about context-free grammars

L where L is a context-free language

Let L be a context-free language

There is a CFG G for L

Then the following TM decides L

On input w

Run TM V from the previous slide on input $\langle G, w \rangle$
If V accepts, accept; else reject

Conclusion: every context-free language L is decidable ✓

Problems about context-free grammars

$$\text{EQ}_{\text{CFG}} = \{\langle G_1, G_2 \rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$$

is not decidable ✗

What's the difference between EQ_{DFA} and EQ_{CFG}?

To decide EQ_{DFA} we minimize both DFAs

But there is no method that, given a CFG or PDA, produces a unique equivalent minimal CFG or PDA

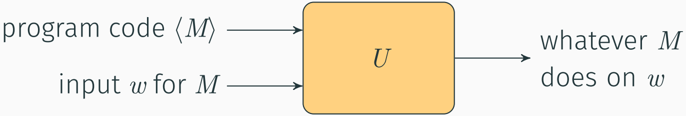
Universal Turing Machine and Undecidability

Turing Machines versus computers

A computer is a machine that manipulates data according to a list of instructions

How does a Turing machine take a program as part of its input?

Universal Turing machine

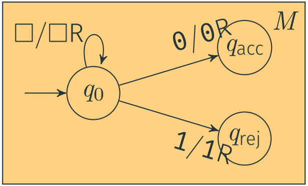


The universal TM U takes as inputs a program M and a string w, and simulates M on w

The program M itself is specified as a TM

Turing machine vs description (executable vs source code)

A Turing machine is
$(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$



A Turing machine can be described by a string $\langle M \rangle$

Turing machine description $\langle M \rangle$

```
(q,qa,qr)(0,1)(0,1,□)
((q,q,□/□R)(q,qa,0/0R)(q,qr,1/1R))
(q)(qa)(qr)
```

Analogy in Python

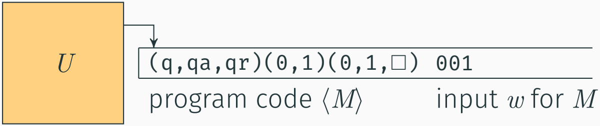
Compiled bytecode

```
2     0 LOAD_GLOBAL       0 (print)
      2 LOAD_CONST        1 ('Hello world')
      4 CALL_FUNCTION     1
      6 POP_TOP
      8 LOAD_CONST        0 (None)
      10 RETURN_VALUE
```

Source code

```
def f(x):
    print("Hello world")
```

Universal Turing machine



(Universal) Turing machine U: on input $\langle M, w \rangle$

Simulate M on input w

If M enters accept state, U accepts

If M enters reject state, U rejects

Acceptance of Turing machines

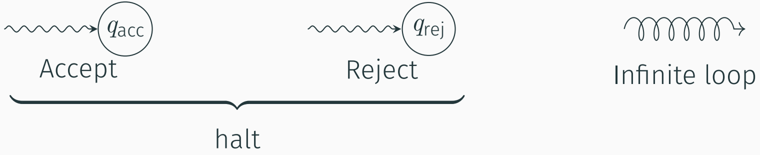
$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$$

U on input $\langle M, w\rangle$ simulates M on input w

M accepts w	M rejects w	M loops on w
$\Downarrow$	$\Downarrow$	$\Downarrow$
U accepts $\langle M, w\rangle$	U rejects $\langle M, w\rangle$	U loops on $\langle M, w\rangle$

TM U recognizes A_{TM} but does not decide A_{TM}

Recognizing versus deciding



The language recognized by a TM M is the set of all inputs that M accepts

A TM decides language L if it recognizes L and halts on every input

A language L is decidable if some TM decides L