Turing Machines

CSCI 3130 Formal Languages and Automata Theory

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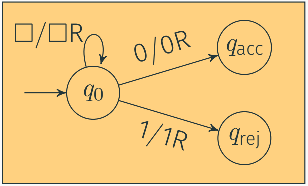
Fall 2021

Chinese University of Hong Kong

Looping

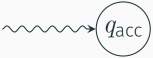
Turing machine may not halt



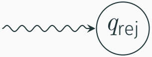
$\Sigma = \{0, 1\}$

input: ε

Inputs can be divided into three types:



Accept



Reject

Infinite loop

Halting

We say M halts on input x if there is a sequence of configurations

$$C_0, C_1, \ldots, C_k$$

C_0 is starting $\quad$ C_i yields C_{i+1} $\quad$ C_k is accepting or rejecting

A TM M is a decider if it halts on every input

A TM M decides a language L if M is a decider and recognizes L

Language L is decidable if it is recognized by a TM that halts on every input

Programming Turing machines: Are two strings equal?

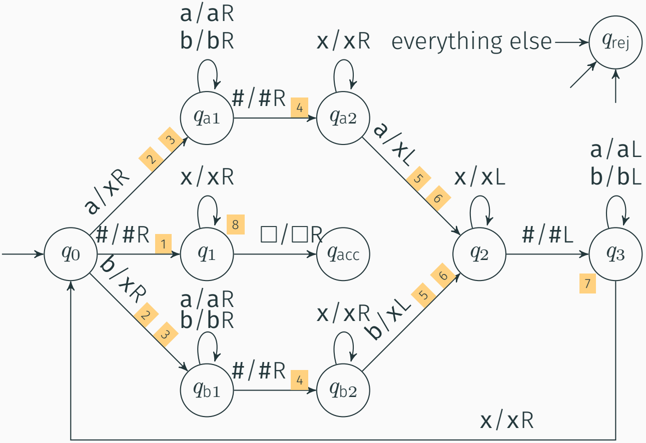
$$L_1 = \{w\#w \mid w \in \{\mathsf{a}, \mathsf{b}\}^*\}$$

Description of Turing Machine

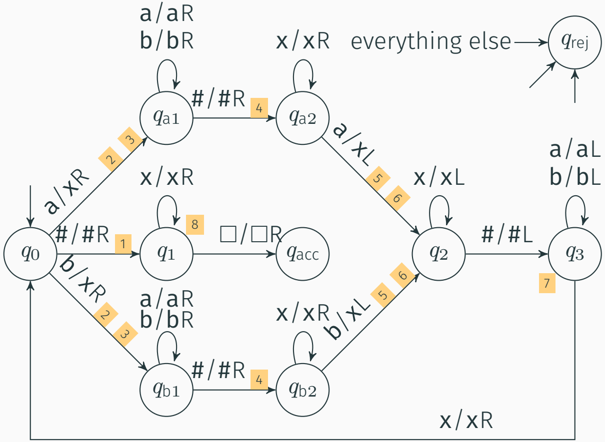
Until you reach #	
Read and remember entry	$\mathsf{x\underline{b}baa\#xbbaa}$
Write x	$\mathsf{xx\underline{b}aa\#xbbaa}$
Move right past # and past all x's	$\mathsf{xxbaa\#x\underline{b}baa}$
If this entry is different, reject	
Write x	$\mathsf{xxbaa\#x\underline{x}baa}$
Move left past # and to right of first x	$\mathsf{xx\underline{b}aa\#xxbaa}$
If you see only x's followed by $\square$, accept	

Programming Turing machines: Are two strings equal?

$$L_1 = \{w\#w \mid w \in \{\mathsf{a}, \mathsf{b}\}^*\}$$



Programming Turing machines: Are two strings equal?



input:

aab#aab

configurations:

q_0 aab#aab

x q_{a1} ab#aab

xa q_{a1} b#aab

xab q_{a1} #aab

xab# q_{a2} aab

xab q_2 #xab

xa q_3 b#xab

x q_3 ab#xab

q_3 xab#xab

x q_0 ab#xab

⋮

$$L_2 = \{\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k \mid ij = k \text{ and } i, j, k > 0\}$$

High level description of TM:

1. For every a:
2. Cross off the same number of b's and c's
3. Uncross the crossed b's (but not the c's)
4. Cross off this a
5. If all a's and c's are crossed off, accept

Example:

1. aabbcccc
2. aa~~bbcc~~cc
3. aabb~~cc~~cc
4. ~~a~~abb~~cc~~cc
5. ~~a~~abb~~cc~~cc
2. ~~a~~a~~bbcccc~~
3. ~~aa~~bb~~cccc~~

$$\Sigma = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\} \qquad \Gamma = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \cancel{\mathsf{a}}, \cancel{\mathsf{b}}, \cancel{\mathsf{c}}, \square\}$$

$$L_2 = \{\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k \mid ij = k \text{ and } i, j, k > 0\}$$

Low-level description of TM:

Scan input from left to right to check it looks like $\mathsf{aa}^*\mathsf{bb}^*\mathsf{cc}^*$

Move the head to the first symbol of the tape

For every a:

- Cross off the same number of b's and c's
- Restore the crossed off b's (but not the c's)
- Cross off this a

If all a's and c's are crossed off, accept

$$L_2 = \{\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k \mid ij = k \text{ and } i, j, k > 0\}$$

Low-level description of TM:

Scan input from left to right to check it looks like $\mathsf{aa}^*\mathsf{bb}^*\mathsf{cc}^*$

Move the head to the first symbol of the tape How?

For every a:

- Cross off the same number of b's and c's How?
- Restore the crossed off b's (but not the c's)
- Cross off this a

If all a's and c's are crossed off, accept

Implementation details:

Move the head to the first symbol of the tape:	
Put a special marker on top of the first a	$\dot{a}abbcccc$
Cross off the same number of b's and c's:	$\dot{a}abbcccc$
Replace b by $\cancel{b}$	$\dot{a}a\cancel{b}bcccc$
Move right until you see a c	$\dot{a}a\cancel{b}bcccc$
Replace c by $\cancel{c}$	$\dot{a}a\cancel{b}b\cancel{c}ccc$
Move left just past the last $\cancel{b}$	$\dot{a}a\cancel{b}\cancel{b}\cancel{c}ccc$
If any uncrossed b's are left, repeat	$\dot{a}a\cancel{b}\cancel{b}\cancel{c}ccc$
	$\dot{a}a\cancel{b}\cancel{b}\cancel{c}\cancel{c}cc$

$$\Sigma = \{a, b, c\} \qquad \Gamma = \{a, b, c, \cancel{a}, \cancel{b}, \cancel{c}, \dot{a}, \cancel{\dot{a}}, \Box\}$$

Programming Turing machines: Element distinctness

$L_3 = \{\#x_1\#x_2 \ldots \#x_m \mid x_i \in \{0, 1\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$

Example: $\#01\#0011\#1 \in L_3$

High-level description of TM:

On input w

For every pair of blocks x_i and x_j in w

Compare the blocks x_i and x_j

If they are the same, reject

Accept

Programming Turing machines: Element distinctness

$L_3 = \{\texttt{\#}x_1\texttt{\#}x_2 \ldots \texttt{\#}x_m \mid x_i \in \{\texttt{0}, \texttt{1}\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$

Low-level desrciption:

0. If input is ε, or has exactly one #, accept
1. Mark the leftmost # as $\dot{\texttt{\#}}$ and move right $\dot{\texttt{\#}}\texttt{01\#0011\#1}$
2. Mark the next unmarked # $\dot{\texttt{\#}}\texttt{01}\dot{\texttt{\#}}\texttt{0011\#1}$

Programming Turing machines: Element distinctness

$$L_3 = \{\texttt{\#}x_1\texttt{\#}x_2\ldots\texttt{\#}x_m \mid x_i \in \{\texttt{0},\texttt{1}\}^* \text{ and } x_i \neq x_j \text{ for every } i \neq j\}$$

3. Compare the two strings to the right of $\dot{\texttt{\#}}$
   If they are equal, reject

   $\dot{\texttt{\#}}\underline{\texttt{01}}\dot{\texttt{\#}}\underline{\texttt{0011}}\texttt{\#1}$

4. Move the right $\dot{\texttt{\#}}$
   If not possible, move the left $\dot{\texttt{\#}}$ to the next $\texttt{\#}$
   and put the right $\dot{\texttt{\#}}$ on the next $\texttt{\#}$
   If not possible, accept

   $\dot{\texttt{\#}}\texttt{01\#0011}\dot{\texttt{\#}}\texttt{1}$

5. Repeat Step 3

   $\dot{\texttt{\#}}\underline{\texttt{01}}\texttt{\#0011}\dot{\texttt{\#}}\underline{\texttt{1}}$

   $\texttt{\#01}\dot{\texttt{\#}}\texttt{0011}\dot{\texttt{\#}}\texttt{1}$

   $\texttt{\#01}\dot{\texttt{\#}}\underline{\texttt{0011}}\dot{\texttt{\#}}\underline{\texttt{1}}$

How to describe Turing Machines

Unlike for DFAs, NFAs, PDAs, we rarely give complete state diagrams of Turing Machines

We usually give a high-level description

unless you're asked for a low-level description or even state diagram

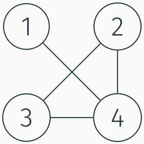
We are interested in algorithms behind the Turing machines

Programming Turing machines: Graph connectivity

$L_4 = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$

How do we feed a graph into a Turing Machine?

How to encode a graph G as a string $\langle G \rangle$?



`(1,2,3,4)((1,4),(2,3),(3,4),(4,2))`

Conventions for describing graphs:

`(`nodes`)(`edges`)`
no node appears twice
edges are pairs `(`first node`,` second node`)`

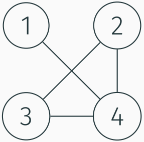
Programming Turing machines: Graph connectivity

$L_3 = \{\langle G \rangle \mid G \text{ is a connected undirected graph}\}$

High-level description:

On input $\langle G \rangle$

0. Verify that $\langle G \rangle$ is the description of a graph
   No node/edge repeats; Edge endpoints are nodes
1. Mark the first node of G
2. Repeat until no new nodes are marked:
   2.1 For each node, mark it if it is adjacent to an already marked node
3. If all nodes are marked, accept; otherwise reject



Programming Turing machines: Graph connectivity

Some low-level details:

0. Verify that $\langle G \rangle$ is the description of a graph

No node/edge repeats: Similar to Element distinctness

Edge endpoints are nodes: Also similar to Element distinctness

1. Mark the first node of G

Mark the leftmost digit with a dot, e.g. 12 becomes $\dot{1}2$

2. Repeat until no new nodes are marked:

2.1 For each node, mark it if it is attached to an already marked node

For every dotted node u and every undotted node v:

Underline both u and v from the node list

Try to match them with an edge from the edge list

If not found, remove underline from u and/or v and try another pair