Notes 22: High dimensional expander

1. Abstract simplicial complex

We want to apply the sampling algorithm based on random walk/Markov chain in the last lecture to other settings (spanning trees, d-paths, d-cliques, etc). Let us generalize those constructions.

Definition 1.1 (Abstract simplicial complex)**.** A set system $Y = (U, \mathcal{F})$ is a ground set U together with a family $\mathcal{F}$ of subsets over U. An abstract simplicial complex is downward closed set system: If $f \in \mathcal{F}$ and $g \subseteq f$, then $g \in \mathcal{F}$.

Abstract simplicial complex in combinatorics was originally proposed to describe the combinatorial structure of a (non-abstract) simplicial complex in algebraic topology. We need not worry about that motivation. Simply think of an abstract simplicial complex as a downward-closed set system.

Definition 1.2 (Level)**.** Level i of an abstract simplicial complex Y is the family of subsets of size i in Y, and is denoted $Y(i) = \{f \in \mathcal{F} \mid |f| = i\}$. The top level $Y(d)$ of Y is the non-empty level with the maximum d.

In the literature, $f \in \mathcal{F}$ of size i is also called a face of dimension $i - 1$. The collection of all such faces is denoted $X(i-1)$ (same as our $Y(i)$). I do not follow the standard terminology of "dimension", since this off-by-one is more confusing than helpful.

Definition 1.3 (Pure)**.** An abstract simplicial complex Y is pure if every face $f \in Y(i)$ is contained in some $g \in Y(d)$ in its top level.

Definition 1.4 (Weight)**.** Weight $w : Y(d) \to \mathbb{R}_+$ assigns positive weights to the maximal faces of a pure abstract simplicial complex Y.

Random walk on pure abstract simplicial complex Y

> Let f_0 be an arbitrary face in the top level $Y(d)$
> For $t = 0, 1, 2, \ldots$
>   Remove an element from f_t uniformly at random to obtain $g_t \in Y(d-1)$
>   Among all $f_{t+1} \supset g_t$, pick the new $f_{t+1} \in Y(d)$ with probability proportional to $w(f_{t+1})$

This is a random walk/Markov chain on a weighted graph with vertex set $Y(d)$, and two nodes are adjacent if they share exactly $d-1$ elements.

An abstract simplicial complex $Y = (U, \mathcal{F})$ with top level $Y(d)$ represents a hypergraph, whose vertex set is U and whose set of hyperedges is $Y(d)$. When the top level is $Y(2)$, we get a graph (and weight is the usual edge weight).

From now on simply call the combinatorial set system a simplicial complex (without "abstract").

2. Inclusion graph

Definition 2.1 (Bipartite inclusion graph)**.** For $0 \leqslant k \leqslant d$, Γ_k has vertex set $Y(k) \cup Y(k-1)$. $t \in Y(k)$ is adjacent to $b \in Y(k-1)$ if $t \supset b$.

Weight trickles down from higher level to lower level via

$$w(b) = \sum_{t \in Y(k),\, t \supset b} w(t) \qquad \text{for } b \in Y(k-1), 0 < k \leqslant d\,. \tag{1}$$

Since the set system is pure, every face at a lower level also gets positive weight.

We recover random walk (up and down transitions) of last lecture, if we set

- distribution π_k over $Y(k)$ to be proportional to the weights: $\pi_k(t) = \dfrac{w(t)}{\sum_{t' \in Y(k)} w(t')}$.
- edge distribution $\mu_k(t, b) = \pi_k(t)/k$

As before, we will also look at $(d+1)$-partite inclusion graph $\Gamma_d \cup \cdots \cup \Gamma_0$.

Positive weight w at the top level plays the same role as last lecture's π, just unnormalized.

Weight at lower level induces distribution that coincides with the bottom marginal π_{k-1} of μ_k:

$$\pi_{k-1}(b) = \sum_{t \sim b} \mu_k(t,b) \propto \sum_{t \sim b} \pi_k(t) \propto \sum_{t \sim b} w(t) = w(b) \ .$$

Therefore weight defined here agrees with (is proportional to) marginals π_k induced from last lecture's random path process $f_d \supset \cdots \supset f_0$ over $\Gamma_d \cup \cdots \cup \Gamma_0$, but unnormalized.

Working with weight w is more convenient than π_k, since we need not worry about normalization. In this case edge weight $w(t,b)$ is $w(t)$. Eq. (1) says $w(b)$ is simply the degree of b under these edge weights. The degree of t is $kw(t) \propto w(t)$.

3. Links

Recall Garland's method decomposed $\tilde{P}_k^\wedge$ into $\sum_b \tilde{P}_b^\wedge$ over $b \in Y(k-1)$, and $P_k^\vee = \sum_b P_b^\vee$.

$\tilde{P}_b^\wedge$ corresponds to transitions in a weighted subgraph $H_b = (S_b, E_b)$, where

$$S_b = \{m \in Y(k) \mid m \supset b\} \qquad E_b = \{(m,m') \in S_b \times S_b \mid m \cup m' \in Y(k+1)\} \ .$$

In the literature, H_b is known as the 1-skeleton of the link of b:

Definition 3.1 (Link)**.** Given a simplicial complex $Y = (U, \mathcal{F})$ and a face $b \in F$, the link of b is $Y_b = (U, \mathcal{F}_b)$, with faces

$$\mathcal{F}_b = \{f \setminus b \mid f \in \mathcal{F},\ f \supseteq b\} \ .$$

$\mathcal{F}_b$ consists of faces g that can extend b to remain in $\mathcal{F}$, so that $g \cup b \in \mathcal{F}$.

Every link Y_b in a pure simplicial complex Y is also a pure simplicial complex.

Definition 3.2 (Skeleton)**.** Given a simplicial complex $Y = (U, \mathcal{F})$, its k-skeleton $(U, \mathcal{F}_k)$ consists of faces in $\mathcal{F}$ of size at most $k+1$.

Think of $(U, \mathcal{F})$ as a hypergraph. 0- and 1-skeletons represent vertices and (non-hyper) edges.

Also, S_b is the 0-skeleton of the link of b.

On one hand, H_b is a pure simplicial complex with weight w induced from the weight of Y by Eq. (1).

On the other hand, H_b is a graph on S_b with edge weight w.

Consider random walk on H_b with edge weights w. It has transition probability

$$P_b(m,m') = \begin{cases} \dfrac{w(m \cup m')}{w(m)} & m \cap m' = b \\ 0 & \text{otherwise} \end{cases} \ .$$

The non-lazy up-walk $\tilde{P}_b^\wedge$ is the random walk P_b on the 1-skeleton H_b scaled down by k:

$$\tilde{P}_b^\wedge = \frac{1}{k} P_b \ ,$$

because both are supported on transitions satisfying $m \cap m' = b$, and for these m and m'

$$\tilde{P}_b^\wedge(m,m') = \frac{w(m \cup m')}{w(m)} \frac{1}{k} = \frac{1}{k} P_b(m,m') \ .$$

Note that P_b is also the non-lazy up-walk in the second layer of the link Y_b. On the other hand, the usual up-walk in the same layer coincides with the lazy random walk on the weighted graph H_b.

Down-walk $P_b^\vee$ has transition probability

$$P_b^\vee(m,m') = \frac{w(m')}{kw(b)} =: \frac{1}{k} \overline{P}_b(m,m') \qquad \text{for } m, m' \in S_b \ .$$

Since $\sum_{m' \in S_b} w(m') = w(b)$ by Eq. (1), $\overline{P}_b$ is the same as the transition probability of a weighted clique over S_b that moves to m' with probability proportional to $w(m')$.

We claimed in last lecture that $\tilde{P}_b^\wedge \preccurlyeq_\Pi P_b^\vee$ when the pure simplicial complex is a matroid. Multiplying both sides by k, this is equivalent to $P_b \preccurlyeq_\Pi \overline{P}_b$.

This is the same as $\lambda_2(P_b) \leqslant 0$, since $\overline{P}_b$ has rank 1 and have all non-trivial eigenvalues 0.

Definition 3.3 (Link expander)**.** A pure simplicial complex Y with weight w is an α-link expander if $\lambda_2(P_b) \leqslant \alpha$ for all $b \in Y(k-1)$ and $0 < k < d-1$.

In this convoluted language, the yet unproved lemma in last lecture becomes:

Lemma 3.4. *If Y is the pure simplicial complex of a matroid of rank d with uniform weight $w = \mathbb{1}$ on $Y(d)$, then Y is a 0-link expander.*

4. Spectra of transition vs normalized adjacency matrix

An undirected graph with adjacency matrix A and (diagonal) degree matrix W has normalized adjacency matrix $\mathcal{A} = W^{-1/2}AW^{-1/2}$. Since $\mathcal{A}$ is symmetric, Courant-Fischer says its k-th largest eigenvalue λ_k is

$$\lambda_k = \max_{S\colon \dim(S)=k} \min_{x \in S\setminus\{0\}} \frac{\langle x, \mathcal{A}x\rangle}{\langle x, x\rangle},$$

where $\langle x, y\rangle$ denotes the inner product $\langle x, y\rangle = x^\top y = \sum_{i\in V} x(i)y(i)$.

The random walk transition matrix $P = W^{-1}A$ is not symmetric, so Courant–Fischer does not apply directly to P. But $P = W^{-1/2}\mathcal{A}W^{1/2}$ is similar to $\mathcal{A}$, which is symmetric, so we can apply Courant–Fischer indirectly using a change of basis via W.

Given $W \succcurlyeq 0$, define positive-semidefinite inner product $\langle x, y\rangle_W = \langle W^{1/2}x, W^{1/2}y\rangle = x^\top W y$.

When $W \succ 0$,

$$\begin{aligned} \frac{\langle x, \mathcal{A}x\rangle}{\langle x, x\rangle} &= \frac{x^\top \mathcal{A}x}{x^\top x} = \frac{x^\top W^{-1/2}AW^{-1/2}x}{x^\top x} = \frac{z^\top A z}{z^\top W z} \qquad (\text{let } z = W^{-1/2}x, \text{ so } x = W^{1/2}z) \\ &= \frac{z^\top W P z}{z^\top W z} = \frac{\langle z, Pz\rangle_W}{\langle z, z\rangle_W}. \end{aligned}$$

Therefore the kth largest eigenvalue λ_k of P is

$$\lambda_k = \max_{S\colon \dim(S)=k} \min_{z \in S\setminus\{0\}} \frac{\langle z, Pz\rangle_W}{\langle z, z\rangle_W}.$$

We will also denote by $\|\cdot\|_W$ the seminorm induced by $\langle\cdot,\cdot\rangle_W$, so that $\|z\|_W^2 = \langle z, z\rangle_W$.

5. Oppenheim's theorem

Oppenheim found a way to translate eigenvalue bound on a higher layer links to that of a lower layer.

Theorem 5.1 (Oppenheim)**.** *Let Y be a pure simplicial complex with weight w. Suppose $\lambda_2(P_b) \leqslant \alpha$ for every $b \in Y(1)$. Also, suppose its 1-skeleton graph $H = (Y(1), Y(2))$ is connected. Then H is an $\frac{\alpha}{1-\alpha}$-expander. Equivalently, the random walk P on H satisfies $\lambda_2(P) \leqslant \frac{\alpha}{1-\alpha}$.*

Applying Oppenheim's theorem inductively, we get:

Corollary 5.2. *Let Y be a pure simplicial complex with weight w. Suppose every link Y_b has a connected 1-skeleton graph. Also, suppose the 1-skeleton graph of every $b \in Y(d-2)$ is an α-expander. Then Y is an $\frac{\alpha}{1-(d-1)\alpha}$ expander.*

Before proving Oppenheim's Theorem 5.1, we first sketch the reasons that the hypotheses of the previous theorem holds for the matroid with uniform weight at the top level.

That every link Y_b is connected is due to the exchangable property of matroid (details omitted).

Given any $b \in Y(n-3)$, the 1-skeleton $H_b = (S_b, E_b)$ of Y_b has adjacency matrix

$$A_b(f, f') = \begin{cases} 1 & \text{if } b \cup f \cup f' \text{ is a spanning tree} \\ 0 & \text{otherwise}w \end{cases}.$$

Edges in b induces three connected components in G. Adding two more edges to these components yields a spanning tree, provided the two edges added are connecting different pairs of components. This partitions the 0-skeleton S_b of Y_b into three sets E_1, E_2, E_3.

The adjacency matrix A_b is of the form

$$A_b = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array}\begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_2 \end{array} \\ \begin{pmatrix} O & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & O & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & O \end{pmatrix} \end{array} = \mathbb{1} - \mathbb{1}_{E_1}\mathbb{1}_{E_1}^\top - \mathbb{1}_{E_2}\mathbb{1}_{E_2}^\top - \mathbb{1}_{E_3}\mathbb{1}_{E_3}^\top .$$

Here $\mathbb{1}$ denotes the all-one matrix of appropriate dimension.

The all-one matrix $\mathbb{1}$ on S_b has rank 1 and nonpositive second eigenvalue, so after subtracting three positive semidefinite matrices $\mathbb{1}_{E_i}\mathbb{1}_{E_i}^\top$ from $\mathbb{1}$, A_b also has nonpositive second eigenvalue by Courant–Fishcer.

Therefore the normalized adjacency matrix of Y_b also has nonpositive second eigenvalue.

Proof of Theorem 5.1. The adjacency matrix A on the empty link $H = (Y(1), Y(2))$ is

$$A(f,g) = \begin{cases} w(f \cup g) & f \cup g \in Y(2) \\ 0 & \text{otherwise} \end{cases} .$$

For $b \in Y(1)$, the adjacency matrix A_b on the link of b is

$$A(f,g) = \begin{cases} w(b \cup f \cup g) & b \cup f \cup g \in Y(3) \\ 0 & \text{otherwise} \end{cases} .$$

We can decompose $A = \sum_{b \in Y(1)} A_b$, because Eq. (1) implies

$$w(f \cup g) = \sum_{b \cup f \cup g \in Y(3)} w(b \cup f \cup g) .$$

The random walk transition $P = W^{-1}A$ on H is similar to the normalized adjacency matrix $\mathcal{A} = W^{-1/2}AW^{-1/2}$, so P and $\mathcal{A}$ have the same spectrum. Similarly the transition $P_b = W_b^{-1}A_b$ on Y_b is similar to the normalized adjacency matrix $P_b = W_b^{-1/2}A_bW_b^{-1/2}$.

Let y be a (right-)eigenvector of P with eigenvalue λ. Then

$$\lambda \|y\|_W^2 = \langle y, Py\rangle_W = \langle y, Ay\rangle = \sum_{b \in Y(1)} \langle y, A_b y\rangle = \sum_{b \in Y(1)} \langle y, P_b y\rangle_{W_b} . \tag{2}$$

Recall that the top (right-)eigenvector of P_b is $\mathbb{1}_{S_b}$, with eigenvalue 1.

Let $\Pi_b^{\|}$ denote projection to the span of $\mathbb{1}_{S_b}$, and $\Pi_b^{\perp}$ denote projection to the orthogonal complement:

$$\Pi_b^{\|}(y) = \frac{\langle y, \mathbb{1}_{S_b}\rangle_W}{\langle \mathbb{1}_{S_b}, \mathbb{1}_{S_b}\rangle_W}\mathbb{1}_{S_b} \quad \text{and} \quad \Pi_b^{\perp}(y) = y - \Pi_b^{\|}(y) .$$

Expand every term in the sum Eq. (2) as

$$y = y_b^{\|} + y_b^{\perp} \quad \text{where} \quad y_b^{\|} = \Pi_b^{\|}(y) = \langle y, \mathbb{1}_{S_b}\rangle_{W_b}\mathbb{1}_{S_b} \quad \text{and} \quad y_b^{\perp} = \Pi_b^{\perp}(y) .$$

Then

$$\langle y, P_b y\rangle_{W_b} = \left\langle y_b^{\|}, P_b y_b^{\|}\right\rangle_{W_b} + \left\langle y_b^{\perp}, P_b y_b^{\perp}\right\rangle_{W_b} , \tag{3}$$

using the fact that

$$\left\langle y_b^{\perp}, P_b y_b^{\|}\right\rangle_{W_b} = \left\langle y_b^{\perp}, P_b \mathbb{1}_{S_b}\right\rangle_{W_b} \langle y, \mathbb{1}_{S_b}\rangle_{W_b} = 0$$

because $P_b\mathbb{1}_{S_b} = \mathbb{1}_{S_b}$, which is W_b-orthogonal to $y_b^{\perp}$.

For the second term in Eq. (3), the assumption that $\lambda_2(P_b) \leqslant \alpha$ implies

$$\left\langle y_b^{\perp}, P_b y_b^{\perp}\right\rangle_{W_b} \leqslant \alpha \|y_b^{\perp}\|_{W_b}^2 = \alpha \sum_{b \in Y(1)} \left(\|y\|_{W_b}^2 - \|y_b^{\|}\|_{W_b}^2\right) = \alpha\|y\|_W^2 - \alpha \sum_{b \in Y(1)} \|y_b^{\|}\|_{W_b}^2 ,$$

where the last equality is

$$\sum_{b \in Y(1)} \|y\|_{W_b}^2 = \sum_{b \in Y(1)} y^\top W_b y = y^\top W y$$

due to

$$W = \mathrm{Diag}(w) = \sum_{b \in Y(1)} \mathrm{Diag}(w_b) = \sum_{b \in Y(1)} W_b \ .$$

For the first term in Eq. (3), since $y_b^{\parallel}$ is a (right-)eigenvector of P_b with eigenvalue 1,

$$\left\langle y_b^{\parallel}, P_b y_b^{\parallel} \right\rangle_{W_b} = \|y_b^{\parallel}\|_{W_b}^2 \ .$$

Therefore Eq. (2) becomes

$$\lambda \|y\|_W^2 \leqslant \alpha \|y\|_W^2 + (1-\alpha) \sum_{b \in Y(1)} \|y_b^{\parallel}\|_{W_b}^2 \ .$$

We have

$$\|y_b^{\parallel}\|_{W_b}^2 = \frac{\langle y, \mathbb{1}_{S_b} \rangle_{W_b}^2}{\|\mathbb{1}_{S_b}\|_{W_b}^2} \ .$$

Note that

$$\|\mathbb{1}_{S_b}\|_{W_b}^2 = \sum_{v \in S_b} w_b(v) = \sum_{v \in S_b} w(b \cup v) = w(b)$$

using Eq. (1), and

$$\langle y, \mathbb{1}_{S_b} \rangle_{W_b} = \sum_{i \in Y(1)} y(i) w_b(i) = \sum_{i \in Y(1)} y(i) w(b \cup i) = (Ay)_b \ .$$

$$\sum_{b \in Y(1)} \|y_b^{\parallel}\|_{W_b}^2 = \sum_{b \in Y(1)} \frac{\langle y, \mathbb{1} \rangle_{W_b}^2}{\|\mathbb{1}_{S_b}\|_{W_b}^2} = \sum_{b \in Y(1)} \frac{(Ay)_b^2}{w(b)} = \langle Ay, W^{-1} Ay \rangle = \langle W^{-1} Ay, W^{-1} Ay \rangle_W$$
$$= \|Py\|_W^2 = \lambda^2 \|y\|_W^2 \ .$$

Hence

$$\lambda y_W^2 \leqslant \alpha \|y\|_W^2 + (1-\alpha)\lambda^2 \|y\|_W^2 \ ,$$

so

$$\lambda - \lambda^2 \leqslant \alpha(1 - \lambda^2) \ .$$

The assumption that the empty link is connected means $\lambda < 1$. Divide both sides by $1 - \lambda$ to get

$$\lambda \leqslant \alpha(1 + \lambda) \qquad \Longrightarrow \qquad \lambda \leqslant \frac{\alpha}{1 - \alpha} \ . \qquad \square$$

Oppenheim's theorem implies

$$\tilde{P}_b^{\wedge} \preccurlyeq_{W_b} P_b^{\vee}$$

for every link b in the simplicial complex of a matroid. To get

$$\tilde{P}_k^{\wedge} \preccurlyeq_{\Pi_k} P_k^{\vee}$$

as claimed in Notes20, we need an analysis similar to the proof of Theorem 5.1.