CSCI5160 Approximation Algorithms Spring 2020
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# Notes 04: Conjugate function

## 1. Convex functions

**Definition 1.1.** A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ on $n$-dimensional Euclidean space is convex if for every $x, y \in \mathbb{R}^n$ and every $0 \leqslant \lambda \leqslant 1$, we have

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

In other words, if we consider the graph of a function, defined as $\{(x, f(x)) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}$, then $f$ is convex if the line segment between any two points of the graph lies above or on the graph.

## 2. Conjugate function

We now define a dual object for every function $f : \mathbb{R}^n \to \mathbb{R}$, called its conjugate.

We have defined dual objects for sets, using support functions. To define a dual object for a function, we want to first turn $f : \mathbb{R}^n \to \mathbb{R}$ into a set.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ (not necessarily convex), its epigraph is $\operatorname{epi} f = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqslant t\}$.

Note that a function is convex if and only if its epigraph is a convex set, as can be easily checked.

The conjugate of a function $f$ is essentially the support function of $\operatorname{epi} f$, "simplified".

The support function of $\operatorname{epi} f$ is $S_{\operatorname{epi} f}(y, s) = \sup\{\langle y, x\rangle + st \mid x \in \mathbb{R}^n, f(x) \leqslant t\}$.

But if $s > 0$, $S_{\operatorname{epi} f}$ says nothing about $f$, because the supremum is $+\infty$ by taking arbitrarily large $t$. If $s = 0$, $S_{\operatorname{epi} f}$ also says nothing about $f$. Only when $s < 0$ does $S_{\operatorname{epi} f}$ capture information about $f$. In this case we always choose $t = f(x)$ in the supremum without changing the outcome. Given any $(y, s)$ with $s < 0$, we can renormalize $(y, s)$ so that $s = -1$. This motivates the following definition.

**Definition 2.1.** Given a function $f : \mathbb{R}^n \to \mathbb{R}$, its conjugate $f^* : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$f^*(y) = \sup\{\langle y, x\rangle - f(x) \mid x \in \mathbb{R}^n\}.$$

Turns out $f^*$ is always convex even when $f$ is not, since it is the pointwise supremum of convex (in this case, affine) functions of $y$.

Under an additional technical assumption, we can indeed recover $f$ as the conjugate of $f^*$.

**Theorem 2.2.** *If $F$ is convex and its epigraph is a closed set, then $f^{**} = f$.*

We will not prove this theorem; see [BV, Exercise 3.39].

In fact $f^{**}$ is the lower semi-continuous envelop of $f$, that is, the largest lower semi-continuous function upper-bounded by $f$. (We will not define semi-continuous here; just think of it as a weaker notion than continuity.)

**Proposition 2.3** (Fenchel inequality)**.** *For any $x, y \in \mathbb{R}^n$, $\langle y, x\rangle \leqslant f^*(y) + f(x)$.*

The proof follows from the definition of conjugate.

Examples of functions and their conjugates:

- *Negative entropy.* $f(x) = x \log x$, defined for $x \geqslant 0$. Then $f^*(y) = \sup_{x \geqslant 0} yx - x \log x$
  The supremum is achieved when $0 = \frac{d}{dx}(yx - x\log x) = y - x(\frac{1}{x}) - \log x \iff x = e^{y-1}$
  Hence $f^*(y) = ye^{y-1} - e^{y-1}(y-1) = e^{y-1}$
- *Strictly convex quadratic form.* $f(x) = \frac{1}{2} x^\top Q x$, where $Q$ is a symmetric positive definite matrix. Then $f^*(y) = \sup_x y^\top x - \frac{1}{2} x^\top Q x$.
  The supremum is achieved when $0 = \nabla(y^\top x - \frac{1}{2} x^\top Q x) = y - Qx \iff x = Q^{-1} y$
  Hence $f^*(y) = y^\top Q^{-1} y - \frac{1}{2}(y^\top Q^{-1\top}) Q (Q^{-1} y) = \frac{1}{2} y^\top Q^{-1} y$
- *Log-sum-exp.* $f(x) = \log(\sum_{1 \leqslant i \leqslant n} e^{x_i})$. [BV, Example 3.25] shows that $f^*(y) = \sum_i y_i \log y_i$, the negative entropy function, restricted to the probability simplex ($y \geqslant 0, \sum_{1 \leqslant i \leqslant n} y_i = 1$).