Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) Consider an undirected, unweighted graph $G = (V, E)$. Suppose $s, t \in V$ are two distinct vertices in $G$. For every vertex $a \in V$, consider starting the random walk at $a$, and let $p(a)$ denote the probability that the random walk reaches $s$ before $t$.

Write down the system of linear equations satisfied by the vector $p \in \mathbb{R}^V$. How is $p(a)$ related to a certain quantity in electric flow that we studied in class?

(2) Consider the barbell graph on $2n$ vertices. It is an undirected and unweighted graph, and is the disjoint union of two non-overlapping cliques, each containing $n$ vertices, plus a single edge that has an endpoint in each clique. Altogether it has $2\binom{n}{2} + 1$ edges. (Look up "barbell graph" on MathWorld for an illustration.)

(a) Show that the edge joining the two cliques has effective resistance 1, while any other edge has effective resistance $\Theta(1/n)$.

(b) Suppose we want to sparsify the barbell graph by randomly sampling $m'$ edges to get a sparse graph $H$. Unlike the algorithm in class, we simply sample $m'$ edges uniformly at random (and not proportional to their effective resistances). Show that whenever $m' = o(n^2)$, with probability approaching 1, the resulting graph $H$ will not satisfy

$$\frac{1}{2} L_G \preccurlyeq L_H \preccurlyeq \frac{3}{2} L_G.$$

This shows that sampling edges uniformly at random will fail to sparsify the graph. *Hint: What is the probability that the inter-clique edge appears in $H$?*

(3) The following question is part of the remaining proof of the Matrix Tree Theorem:

Prove that if $F \subseteq E$ is a spanning tree of an undirected, unweighted graph $G$ on $n$ vertices, then

$$\det(B_F B_F^T) = n.$$

Here $B_F$ is the $F \times V$ edge-vertex incidence matrix, so that (after arbitrarily orienting each edge)

$$B_F(e, v) = \begin{cases} 1 & \text{if } e = (u, v) \\ -1 & \text{if } e = (v, u) \\ 0 & \text{if } v \notin e \end{cases}.$$

(4) Let $G$ be an undirected, unweighted, connected graph. Suppose the effective resistance of every edge is at most $1/k$ (by the effective resistance of an edge $e = (a, b)$, we mean the effective resistance of its endpoints $a$ and $b$). Prove that $G$ is $k$-edge connected, that is, $E(S, \overline{S}) \geqslant k$ for every nonempty $S \subsetneq V$.

*Hint: Prove the contrapositive: Suppose $S$ is a cut such that $E(S, \overline{S}) = k$, show that any edge across this cut has effective resistance at least $1/k$. The original definition of effective resistance will help you. The short proof is about 10 sentences long.*

(5) (a) Let $G = (V, E)$ be an undirected, unweighted, $d$-regular graph. Suppose $\psi_1, \ldots, \psi_n$ are the orthonormal eigenvectors of its normalized Laplacian matrix. Given a positive integer $k$, consider the following $k$-dimensional spectral embedding: For every vertex $v \in V$, let $\Psi(v) = (\psi_1(v), \ldots, \psi_k(v)) \in \mathbb{R}^k$. Show that the vectors $\{\Psi(v)\}_{v \in V}$ are in isotropic position.

(b) Let $G$ be as above, and define the Rayleigh quotient for an embedding $\Psi : V \to \mathbb{R}^k$ as

$$R(\Psi) \stackrel{\text{def}}{=} \frac{\sum_{(u,v) \in E} \|\Psi(u) - \Psi(v)\|_2^2}{\sum_v d \|\Psi(v)\|_2^2} .$$

Suppose $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ are the corresponding eigenvalues of the normalized Laplacian. Show that the $k$-dimensional spectral embedding $\Psi$ in part (a) has Rayleigh quotient at most $\lambda_k$.