Decidability

CSCI 3130 Formal Languages and Automata Theory

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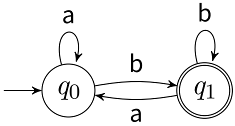
Problems about automata

Does



accept input abb?

We can formulate this question as a language

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Is A_{DFA} decidable?

One possible way to encode a DFA $D = (Q, \Sigma, \delta, q_0, F)$ and input w

$$\texttt{(}\underbrace{\texttt{(q0,q1)}}_{Q}\underbrace{\texttt{(a,b)}}_{\Sigma}\underbrace{\texttt{((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))}}_{\delta}\underbrace{\texttt{(q0)}}_{q_0}\underbrace{\texttt{(q1)}}_{F}\texttt{)(}\underbrace{\texttt{abb}}_{w}\texttt{)}$$

Problems about automata

$$A_{\textsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Pseudocode:

On input $\langle D, w\rangle$, where $D = (Q, \Sigma, \delta, q_0, F)$

Set $q \leftarrow q_0$
For $i \leftarrow 1$ to length(w)
    $q \leftarrow \delta(q, w_i)$
If $q \in F$ accept, else reject

TM description:

On input $\langle D, w\rangle$, where D is a DFA, w is a string

Simulate D on input w
If simulation ends in an accept state, accept; else reject

Problems about automata

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Turing machine details:

Check input is in correct format
(Transition function is complete, no duplicate transitions)

Perform simulation:

```
((q̇0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(ȧbb)
((q̇0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(aḃb)
((q0,q̇1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(abḃ)
((q0,q̇1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(abb)̇
```

Problems about automata

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Turing machine details:

Check input is in correct format
(Transition function is complete, no duplicate transitions)

Perform simulation: (very high-level)

Put markers on start state of D and first symbol of w
Until marker for w reaches last symbol:
    Update both markers
If state marker is on accepting state, accept; else reject

Conclusion: A_{DFA} is decidable

Acceptance problems about automata

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\} \quad ✓$$

$$A_{\mathsf{NFA}} = \{\langle N, w\rangle \mid N \text{ is an NFA that accepts input } w\}$$

$$A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$$

Which of these is decidable?

Acceptance problems about automata

$$A_{\text{NFA}} = \{\langle N, w\rangle \mid N \text{ is an NFA that accepts input } w\}$$

The following TM decides A_{NFA}:
On input $\langle N, w\rangle$ where N is an NFA and w is a string

Convert N to a DFA D using the conversion procedure from Lecture 3
Run TM M for A_{DFA} on input $\langle D, w\rangle$
If M accepts, accept; else reject

Conclusion: A_{NFA} is decidable ✓

Acceptance problems about automata

$$A_{\text{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$$

The following TM decides A_{REX}

On input $\langle R, w\rangle$, where R is a regular expression and w is a string

Convert R to an NFA N using the conversion procedure from Lecture 4

Run the TM for A_{NFA} on input $\langle N, w\rangle$

If N accepts, accept; else reject

Conclusion: A_{REX} is decidable ✓

Other problems about automata

$$\mathsf{MIN}_{\mathsf{DFA}} = \{\langle D \rangle \mid D \text{ is a minimal DFA}\}$$

$$\mathsf{EQ}_{\mathsf{DFA}} = \{\langle D_1, D_2 \rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$$

$$E_{\mathsf{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) \text{ is empty}\}$$

Which of the above is decidable?

Other problems about automata

$$\text{MIN}_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a minimal DFA}\}$$

The following TM decides MIN_{DFA}
On input $\langle D \rangle$, where D is a DFA

Run the DFA minimization algorithm from Lecture 7
If every pair of states is distinguishable, accept; else reject

Conclusion: MIN_{DFA} is decidable ✓

Other problems about automata

$$\mathsf{EQ}_{\mathsf{DFA}} = \{\langle D_1, D_2\rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$$

The following TM decides $\mathsf{EQ}_{\mathsf{DFA}}$
On input $\langle D_1, D_2\rangle$, where D_1 and D_2 are DFAs

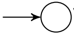
Run the DFA minimization algorithm from Lecture 7 on D_1 to obtain a minimal DFA D_1'
Run the DFA minimization algorithm from Lecture 7 on D_2 to obtain a minimal DFA D_2'
If $D_1' = D_2'$, accept; else reject

Conclusion: $\mathsf{EQ}_{\mathsf{DFA}}$ is decidable ✓

Other problems about automata

$$E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) \text{ is empty}\}$$

The following TM T decides E_{DFA}
On input $\langle D \rangle$, where D is a DFA

Run the TM S for EQ_{DFA} on input $\langle D,$  $\rangle$
If S accepts, T accepts; else T rejects

Conclusion: E_{DFA} is decidable ✓

Problems about context-free grammars

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$$

L where L is a context-free language

$$\mathsf{EQ}_{\mathsf{CFG}} = \{\langle G_1, G_2\rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$$

Which of the above is decidable?

Problems about context-free grammars

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$$

The following TM V decides A_{CFG}

On input $\langle G, w\rangle$, where G is a CFG and w is a string

Eliminate the ε- and unit productions from G

Convert G into Chomsky Normal Form G'

Run Cocke–Younger–Kasami algorithm on $\langle G', w\rangle$

If the CYK algorithm finds a parse tree, V accepts; else V rejects

Conclusion: A_{CFG} is decidable ✓

Problems about context-free grammars

L where L is a context-free language

Let L be a context-free language
There is a CFG G for L

The following TM decides L
On input w

Run TM V from the previous slide on input $\langle G, w \rangle$
If V accepts, accept; else reject

Conclusion: every context-free language L is decidable ✓

Problems about context-free grammars

$\mathsf{EQ}_{\mathsf{CFG}} = \{\langle G_1, G_2\rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$
is not decidable ✗

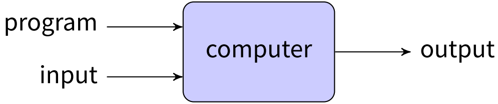
What's the difference between $\mathsf{EQ}_{\mathsf{DFA}}$ and $\mathsf{EQ}_{\mathsf{CFG}}$?

To decide $\mathsf{EQ}_{\mathsf{DFA}}$ we minimize both DFAs

But there is no method that, given a CFG or PDA, produces a unique equivalent minimal CFG or PDA

Universal Turing Machine and Undecidability

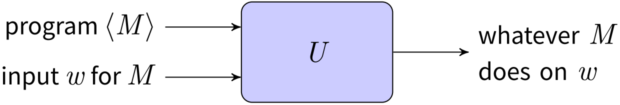
Turing Machines versus computers



A computer is a machine that manipulates data according to a list of instructions

How does a Turing machine take a program as part of its input?

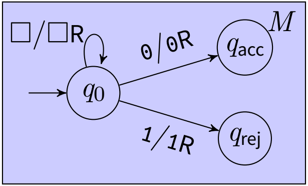
The universal Turing machine



The universal TM U takes as inputs a program M and a string x, and simulates M on w

The program M itself is specified as a TM

Turing machines as strings



A Turing machine is
$(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$

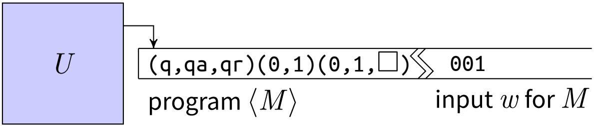
This Turing machine can be described by the string

$\langle M \rangle =$ `(q,qa,qr)(0,1)(0,1,□)`

`((q,q,□/□R)(q,qa,0/0R)(q,qr,1/1R))`

`(q)(qa)(qr)`

The universal Turing machine



U on input $\langle M, w \rangle$:

Simulate M on input w
If M enters accept state, U accepts
If M enters reject state, U rejects

Acceptance of Turing machines

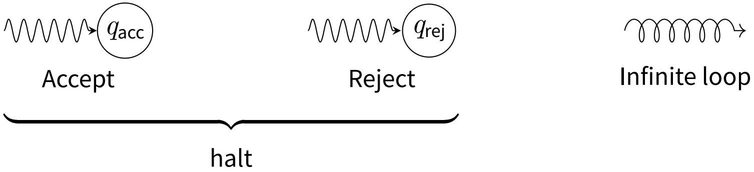
$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$$

U on input $\langle M, w\rangle$ simulates M on input w

M accepts w	M rejects w	M loops on w
$\Downarrow$	$\Downarrow$	$\Downarrow$
U accepts $\langle M, w\rangle$	U rejects $\langle M, w\rangle$	U loops on $\langle M, w\rangle$

TM U recognizes A_{TM} but does not decide A_{TM}

Recognizing versus deciding



The language recognized by a TM M is the set of all inputs that M accepts

A TM decides language L if it recognizes L and halts on every input

A language L is decidable if some TM decides L