Irregular Languages

CSCI 3130 Formal Languages and Automata Theory

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Non-regular languages

Are there irregular languages?

Candidate from last lecture:

$$L = \{0^n 1 0^n 1 \mid n \geqslant 0\}$$

(duplicate of language of $0^* 1 = \{1, 01, 001, 0001, \ldots\}$)

Non-regular languages

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Candidate from last lecture:

$$L = \{0^n 1 0^n 1 \mid n \geqslant 0\}$$

(duplicate of language of $0^*1 = \{1, 01, 001, 0001, \ldots\}$)

Why do we believe it is irregular?

Seems to require a “DFA” with infinitely many states

After reading the first half, need to remember number of zeros so far

11, 0101, 001001, 00010001, …

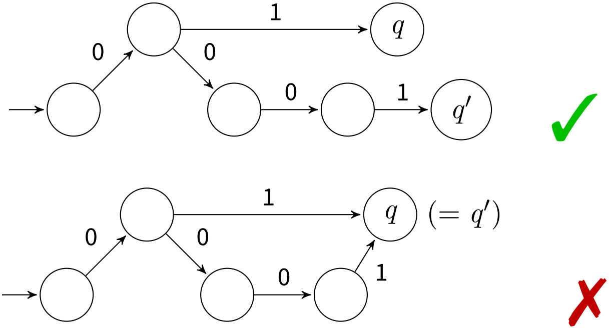
Infinitely many possibilities

Let’s formally prove this intuition

Distinct states for 01 and 0001

Claim

If a deterministic automaton accepts $L = \{0^n 1 0^n 1 \mid n \geqslant 0\}$, the state q it reaches upon reading 01 must be different from the state q' it reaches upon reading 0001



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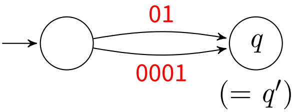
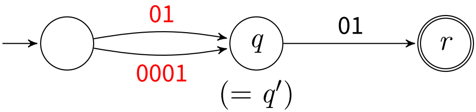


✗

Why not?

Reason: after going to q, if it reads 01 and reaches r …

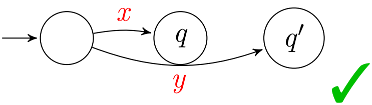


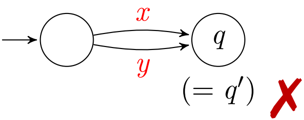
If r is not accepting, it rejects 0101 ✗

If r is accepting state, it accepts 000101 ✗

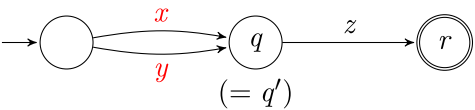
General case: distinguishable strings

If a deterministic automaton accepts L, if there are strings x and y such that $xz \in L$ but $yz \notin L$, then the automaton must be in two different states upon reading x and y





Reason:



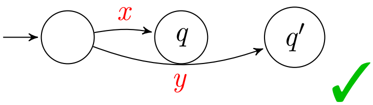
If r is not accepting, it rejects xz ✗

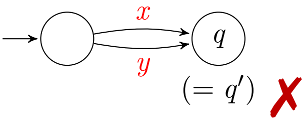
If r is accepting state, it accepts yz ✗

Distinguishable strings

x and y are distinguishable by L if for some string z,
we have $xz \in L$ and $yz \notin L$ (or the other way round)

If x and y are distinguishable by L, any deterministic automaton accepting L must reach different states upon reading x and y

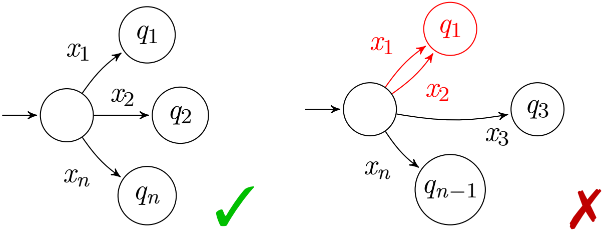




Requires many states

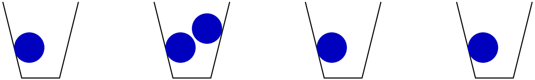
Strings $x_1, \dots, x_n$ are called pairwise distinguishable by L if every pair x_i and x_j are distinguishable by L, for any $i \neq j$.

If strings $x_1, \dots, x_n$ are pairwise distinguishable by L, any deterministic automaton accepting L must have at least n states



Pigeonhole principle

If you put 5 balls into 4 bins,
then (at least) two balls end up in the same bin



More generally

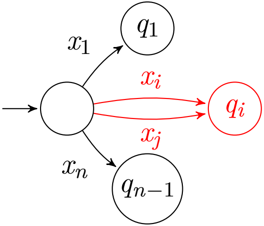
If you put n balls into (at most) $n - 1$ bins,
then (at least) two balls end in the same bin

Pigeonhole principle

Requires many states

If strings $x_1, \ldots, x_n$ are pairwise distinguishable by L, any deterministic automaton accepting L must have at least n states

Otherwise:



If there are (at most) $n-1$ states, by pigeonhole principle, two different strings x_i and x_j must end up at the same state, but:

If x_i and x_j are distinguishable by L, any deterministic automaton accepting L must reach different states upon reading x_i and x_j ✗

0^n10^n1 is not regular

Suffices find an infinitely sequence of strings that are pairwise distinguishable by $L = \{0^n10^n1 \mid n \geqslant 0\}$

After reading the first half, need to remember number of zeros so far
11, 0101, 001001, 00010001, …

1, 01, 001, 0001, . . . are pairwise distinguishable by L

Why are 0^i1 and 0^j1 distinguishable by L? $(i \neq j)$

0^n10^n1 is not regular

Suffices find an infinitely sequence of strings that are pairwise distinguishable by $L = \{0^n10^n1 \mid n \geqslant 0\}$

After reading the first half, need to remember number of zeros so far
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1, 01, 001, 0001, . . . are pairwise distinguishable by L

Why are 0^i1 and 0^j1 distinguishable by L? $(i \neq j)$

Take $z = 0^i1$
$0^i10^i1 \in L$ $0^j10^i1 \notin L$

Which of these are (ir)regular?

$$L_1 = \{x \mid x \text{ has the same number of 0s and 1s}\}$$
$$L_2 = \{0^n 1^m \mid n > m \geqslant 0\}$$
$$L_3 = \{x \mid x \text{ has the same number of patterns 01 and 11}\}$$
$$L_4 = \{x \mid x \text{ has the same number of patterns 01 and 10}\}$$
$$L_5 = \{x \mid x \text{ has a different number of 0s and 1s}\}$$

L_1 : same number of 0s and 1s

Why does it require infinitely many states to accept?

L_1 : same number of 0s and 1s

Why does it require infinitely many states to accept?

Need to remember number of 0s (or 1s) read so far

ε, 0, 00, 000, $\ldots$ are pairwise distinguishable by L_1

Why are 0^i and 0^j distinguishable by L_1? $(i \neq j)$

L_1 : same number of 0s and 1s

Why does it require infinitely many states to accept?

Need to remember number of 0s (or 1s) read so far

ε, 0, 00, 000, $\ldots$ are pairwise distinguishable by L_1

Why are 0^i and 0^j distinguishable by L_1? $(i \neq j)$

Take $z = 1^i$

$0^i 1^i \in L_1 \qquad 0^j 1^i \notin L_1$

$L_2 = \{0^n1^m \mid n > m\}$

Like L_1, need to remember number of 0s read so far

ε, 0, 00, 000, $\ldots$ are pairwise distinguishable by L_2

Why are 0^i and 0^j distinguishable by L_2? $\quad (i > j)$

$L_2 = \{0^n 1^m \mid n > m\}$

Like L_1, need to remember number of 0s read so far

ε, 0, 00, 000, $\ldots$ are pairwise distinguishable by L_2

Why are 0^i and 0^j distinguishable by L_2? $\quad (i > j)$

Take $z = 1^{i-1}$

$$0^i 1^{i-1} \in L_2 \qquad 0^j 1^{i-1} \notin L_2$$

L_3 : same number of 01s and 11s

Need to remember the number of 01s read so far

ε, 01, 0101, 010101, $\ldots$ are pairwise distinguishable by L_3

Why are $(01)^i$ and $(01)^j$ distinguishable by L_3? $(i > j)$

L_3 : same number of 01s and 11s

Need to remember the number of 01s read so far

ε, 01, 0101, 010101, $\ldots$ are pairwise distinguishable by L_3

Why are $(01)^i$ and $(01)^j$ distinguishable by L_3? $\quad (i > j)$

Take $z = 1^i$

$(01)^i 1^i \in L_3 \qquad (01)^j 1^i \notin L_3$

Example: 010101111 $\quad (i = 3)$

L_4 : same number of 01s and 10s

ε, 01, 0101, 010101, . . . are pairwise distinguishable by L_4

Why are $(01)^i$ and $(01)^j$ distinguishable by L_4? $(i > j)$

Take $z = (10)^i$

$(01)^i(10)^i \in L_4 \qquad (01)^j(10)^i \notin L_4$

Example: 010101101010 $(i = 3)$

L_4 : same number of 01s and 10s

~~ε, 01, 0101, 010101, $\ldots$ are pairwise distinguishable by L_4~~

~~Why are $(01)^i$ and $(01)^j$ distinguishable by L_4? $(i > j)$~~

~~Take $z = (10)^i$~~

~~$(01)^i(10)^i \in L_4$~~ ~~$(01)^j(10)^i \notin L_4$~~

Example: 010101101010 $(i = 3)$

In fact, $(01)^j(10)^i \in L_4$ because there are as many 01 as 10

In fact, L_4 is regular (see Week 2 tutorial)

L_5 : different number of 0s and 1s

Is L_5 irregular?

L_5 : different number of 0s and 1s

Is L_5 irregular?

Yes

If L_5 were regular, then so is

$$\overline{L_5} = L_1 = \{x \mid x \text{ has the same number of 0s and 1s}\}$$

But we saw that L_1 is irregular, therefore so is L_5

An exercise

$L_6 =$ lang. of properly nested strings of parentheses $\qquad \Sigma = \{(,)\}$

(), (()), ()() are in L_6

(,),)(are not

Exercise: show that L_6 is irregular

What does it mean?

An exercise

$L_6 =$ lang. of properly nested strings of parentheses $\qquad \Sigma = \{(,)\}$

(), (()), ()() are in L_6

(,),)(are not

Exercise: show that L_6 is irregular

What does it mean?

Language = computational problem

DFA = machine with finite memory

L_6 is irregular ⇒ checking whether (arbitrarily long) strings are properly nested requires unbounded amount of memory