Dynamic Programming 2: Optimal BST

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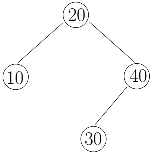
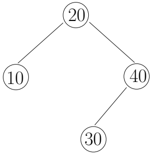
Designing a dynamic programming algorithm, in general, requires discovering a **recursive structure** of the underlying problem. Next, we will illustrate this through the **optimal BST problem**.

Review: Binary Search Tree (BST)



- Each node stores a **key**.
- The key of an internal node u is **larger** than any key in its **left** subtree, and **smaller** than any key in its **right** subtree.

Review: Binary Search Tree (BST)



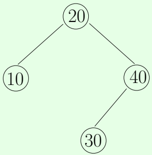
- The **level** of a node u in a BST T — denoted as $level_T(u)$ — equals the number of edges on the path from the root to u.
  - The level of the root is 0.
- The **depth** of a tree is the maximum level of the nodes in the tree.
- Searching for a node u incurs cost proportional to $1 + level_T(u)$.
  - How many nodes do you need to access to search for node 10, 20, 30, and 40, respectively?

Let S be a set of n integers.
We know that a balanced BST on S has depth $O(\log n)$.
This is good if we assume that all the integers in S are searched with **equal probabilities**.

In practice, not all keys are equally important: some are searched **more often than others**. This gives rise to an interesting question:

> If we know the search frequencies of the integers in S, how to build a better BST to minimize the average search cost?

Example:



Suppose that we know the frequencies of 10, 20, 30, and 40 are 40%, 15%, 35%, and 10%, respectively. Then, the average cost of searching for a key in the BST equals:

$$\begin{aligned} & freq(10) \cdot cost(10) + freq(20) \cdot cost(20) + \\ & freq(30) \cdot cost(30) + freq(40) \cdot cost(40) \\ = \ & 40\% \cdot 2 + 15\% \cdot 1 + 35\% \cdot 3 + 10\% \cdot 2 \\ = \ & 2.2 \end{aligned}$$

where *freq*(k) denotes the search frequency of key k, and *cost*(k) denotes the cost of searching for k in the tree.

The Optimal BST Problem

Input:

- A set S of n integers: $\{1, 2, ..., n\}$;
- An array W where $W[i]$ $(1 \le i \le n)$ stores a positive integer weight.

Output:

A BST T on S with the smallest **average cost**:

$$avgcost(T) = \sum_{i=1}^{n} W[i] \cdot cost_T(i).$$

where $cost_T(i) = 1 + level_T(i)$ is the number of nodes accessed to find the key i in T.

> **Think:** here we consider that the keys are $1, 2, ...n$, respectively; do we lose any generality?

A Slightly More General Problem

We will solve a more general version of the problem.

Input:

- S and W same as before;
- Integers a, b satisfying $1 \leq a \leq b \leq n$.

Output:

A BST T on $\{a, a+1, ..., b\}$ with the smallest **average cost**:

$$avgcost(T) = \sum_{i=a}^{b} W[i] \cdot cost_T(i).$$

where $cost_T(i) = 1 + level_T(i)$ is the number of nodes accessed to find the key i in T.

As mentioned, an important step in designing a dynamic programming algorithm is to figure out the **recursive structure** of the underlying problem. Typically, this involves three steps:

1. identify **all** the possible options for the "**first**" choice;
2. **conditioned on** the first choice, find the optimal solution;
3. take the first choice that leads to the **overall best** solution.

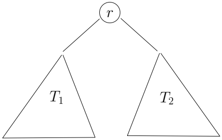
Next, we will explain how to do so for the optimal BST problem.

1. Find all the Options for the First Choice

First Choice: Key at the root of T?

Clearly, we have $b - a + 1$ options: we can put $a, a + 1, ..., $ or b as the key at the root.

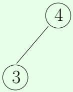
Suppose that we put r as the key at the root for some $r \in [a, b]$. Then, its left subtree must be a BST T_1 on $S_1 = \{a, ..., r - 1\}$, and its right subtree must be a BST T_2 on $S_2 = \{r + 1, ..., b\}$.



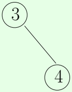
Example: $S = \{1, 2, 3, 4\}$; $W = (40, 15, 35, 10)$.

Consider the option of putting 2 at the root. The left subtree must contain just a single leaf with the key 1.

The right subtree, on the other hand, has two choices:

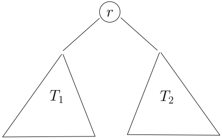


or



2. Conditioned on the First Choice, Find the Optimal Solution:

Put r at the root of T. Next, we will show that, to minimize the average cost of T, we should choose the best trees for T_1 and T_2.



$$
\begin{aligned}
& avgcost(T) \\
= \; & \sum_{i=a}^{b} W[i] \cdot cost_T(i) = \sum_{i=a}^{b} W[i] \cdot (1 + level_T(i)) \\
= \; & \left(\sum_{i=a}^{b} W[i] \right) + \sum_{i=a}^{b} W[i] \cdot level_T(i) \\
= \; & \left(\sum_{i=a}^{b} W[i] \right) + \left(\sum_{i=a}^{r-1} W[i] \cdot level_T(i) \right) + \left(\sum_{i=r+1}^{b} W[i] \cdot level_T(i) \right)
\end{aligned}
$$

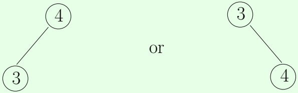
(Continuing on the next slide)

$$\begin{aligned}
&= \left(\sum_{i=a}^{b} W[i]\right) + \left(\sum_{i=a}^{r-1} W[i] \cdot (1 + level_{T_1}(i))\right) + \\
&\quad \left(\sum_{i=r+1}^{b} W[i] \cdot (1 + level_{T_2}(i))\right) \\
&= \left(\sum_{i=a}^{b} W[i]\right) + \left(\sum_{i=a}^{r-1} W[i] \cdot cost_{T_1}(i)\right) + \left(\sum_{i=r+1}^{b} W[i] \cdot cost_{T_2}(i)\right) \\
&= \left(\sum_{i=a}^{b} W[i]\right) + avgcost(T_1) + avgcost(T_2)
\end{aligned}$$

Clearly, we should minimize $avgcost(T_1)$ and $avgcost(T_2)$, namely, building optimal BSTs on S_1 and S_2, recursively.

Example: $S = \{1, 2, 3, 4\}$; $W = (40, 15, 35, 10)$.

Consider the option of putting 2 at the root. As mentioned, the right subtree has two choices:



We know from the above discussion that the right subtree should be an optimal BST on $\{3, 4\}$. Which of the above two choices is optimal on $\{3, 4\}$?

The answer is the second one: it has an average cost of $35 \cdot 1 + 10 \cdot 2 = 55$.

Define *optavg*(a, b) as

- 0, if $a > b$;
- the smallest average cost of a BST on $\{a, a+1, ..., b\}$, otherwise.

Define *optavg*$(a, b \mid r)$ as the optimal average cost of a BST, **on condition that** the BST has r as the key of the root.

The previous discussion has essentially proved:

$$\begin{aligned} & optavg(a, b \mid r) \\ = & \left(\sum_{i=a}^{b} W[i]\right) + optavg(a, r-1) + optavg(r+1, b). \end{aligned}$$

Example: $S = \{1, 2, 3, 4\}$; $W = (40, 15, 35, 10)$.

Consider the option of putting 2 at the root.

$$\begin{aligned} & optavg(1, 4 \mid 2) \\ = \; & \left(\sum_{i=1}^{4} W[i]\right) + optavg(1, 1) + optavg(3, 4) \\ = \; & 100 + 40 + 55 = 195. \end{aligned}$$

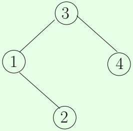
Hence, **if we want to put 2 at the root**, the best BST we can construct has average cost 195.

3. Selecting the Best First Choice: The best choice for r is the one that leads to the smallest average cost, namely:

$$\begin{aligned} & optavg(a, b) \\ = \; & \min_{r=a}^{b} optavg(a, b \mid r) \\ = \; & \left(\sum_{i=a}^{b} W[i] \right) + \min_{r=a}^{b} \Big\{ optavg(a, r-1) + optavg(r+1, b) \Big\}. \end{aligned}$$

This is the recursive structure of the problem.

Example: $S = \{1, 2, 3, 4\}$; $W = (40, 15, 35, 10)$.
The optimal tree is actually:



$$
\begin{aligned}
optavg(1,4) &= optavg(1,4 \mid 3) \\
&= \left(\sum_{i=1}^{4} W[i]\right) + optavg(1,2) + optavg(4,4) \\
&= 100 + optavg(1,2) + 10 = 110 + optavg(1,2) \\
&= 110 + optavg(1,2 \mid 1) \\
&= 110 + \left(\sum_{i=1}^{2} W[i]\right) + optavg(1,0) + optavg(2,2) \\
&= 110 + 55 + 0 + 15 = 180.
\end{aligned}
$$

Putting Everything Together

We have converted the optimal BST problem into the following problem:

Input: An array W of n integers.
Output Compute $optavg(1, n)$ where for any $a, b \in [1, n]$:

$$optavg(a,b) = \begin{cases} 0, \text{ if } a > b \\ \left(\sum_{i=a}^{b} W[i]\right) + \min_{r=a}^{b}\left\{optavg(a, r-1) + optavg(r+1, b)\right\} \\ \text{otherwise} \end{cases}$$

This is precisely the problem we studied in the previous lecture! Recall that with dynamic programming, we can compute $optavg(1, n)$ in $O(n^3)$ time.

Strictly speaking, there is one more step: although we have calculated $optavg(1, n)$, we still have not produced the optimal BST yet!

This is, in fact, rather trivial — you can do so in $O(n)$ time after computing $optavg(1, n)$ with dynamic programming. This will be left as a regular exercise.