Divide and Conquer

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In this lecture, we will discuss the **divide and conquer** technique for designing algorithms with strong performance guarantees. Our discussion will be based on the following problems:

1. Sorting (a review of merge sort)
2. Counting inversions
3. Dominance counting
4. Matrix multiplication

> We will focus on the ideas most relevant to illustrating divide and conquer, and will not try very hard to attain the fastest possible running time. On some problems, improving the running time makes interesting exercises, as will be duly mentioned.

The **thinking** behind divide and conquer:

> Divide the problem into smaller parts. Do we gain anything if those parts have been settled? In particular, can the results of those parts be **combined** efficiently?

Sorting

Sorting

Problem: Given an array A of n distinct integers, produce another array where the same integers have been arranged in ascending order.

Thinking:

- **Divide**: Let A_1 the array containing the first $\lceil n/2 \rceil$ elements of A, and A_2 be the array containing the other elements of A.
  Sort A_1 and A_2 recursively.
- **What do we gain?** It suffices to merge the two sorted arrays A_1, A_2 into an overall ascending order. This can be done in $O(n)$ time.

This is the merge sort algorithm.

Sorting

Running Time: Let $f(n)$ denote the worst-case cost of the algorithm on an array of size n. Then:

$$f(n) \leq 2 \cdot f(\lceil n/2 \rceil) + O(n)$$

which gives $f(n) = O(n \log n)$.

Counting Inversions

Counting Inversions

Let: A = an array of n distinct integers.

An **inversion** is a pair of (i, j) such that

- $1 \le i < j \le n$, and
- $A[i] > A[j]$.

Example: Consider $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$.
Then $(1, 2)$ is an inversion because $A[1] = 10 > A[2] = 3$. So are $(1, 3), (3, 4), (4, 5)$, and so on.
There are in total 29 inversions.

Think: How many inversions can there be in the worst case?
Answer: $\binom{n}{2} = \Theta(n^2)$.

Counting Inversions

Problem: Given an array A of n distinct integers, count the number of inversions.

Trivial: $O(n^2)$ time.
We will do in the class: $O(n \log^2 n)$ time.
You will do as an exercise: $O(n \log n)$ time.

Counting Inversions

Thinking:

- **Divide**: Let A_1 the array containing the first $\lceil n/2 \rceil$ elements of A, and A_2 be the array containing the other elements of A.

  Solve the "counting inversions" problem recursively on A_1 and A_2, respectively. By doing so, we have already obtained the number m_1 of inversions in A_1, and similarly, the number m_2 for A_2.

- **What do we gain?**

  It remains to count the number of **crossing inversions** (i, j) where i is in A_1 and j in A_2.

  $\Rightarrow$

  The relative ordering **within** A_1 no longer matters! Same for A_2!

Counting Inversions

A_1 = the array containing the first $\lceil n/2 \rceil$ elements of A
A_2 = the array containing the other elements of A.

Sort A_1 and A_2.
For each element in A_1, find out how many crossing inversions it produces using **binary search**.

> **Example (cont.):** $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$.
> $A_1 = (2, 3, 8, 9, 10)$, $A_2 = (1, 4, 5, 6, 7)$
>
> Element 2 produces 1 crossing inversion
> Element 3 produces 1, too.
> Elements 8, 9, and 10 each produces 5.
>
> - **Think:** How to obtain each count with binary search?

In total, $n/2$ binary searches are performed, which takes $O(n \log n)$ time.

Counting Inversions

Running Time: Let $f(n)$ denote the worst-case cost of the algorithm on an array of size n. Then:

$$f(n) \leq 2 \cdot f(\lceil n/2 \rceil) + O(n \log n)$$

which gives $f(n) = O(n \log^2 n)$.

Dominance Counting

Dominance Counting

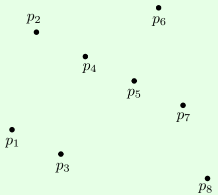
Denote by $\mathbb{N}$ the set of integers.
Given a point p in two-dimensional space $\mathbb{N}^2$, denote by $p[1]$ and $p[2]$ its x- and y-coordinate, respectively.

Given two distinct points p and q, we say that q **dominates** p if $p[1] \leq q[1]$ and $p[2] \leq q[2]$; see the figure below:

Dominance Counting

Let P be a set of n points in $\mathbb{N}^2$. Find, for each point $p \in P$, the number of points in P that are dominated by p.

Example:



We should output: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2), (p_5, 2), (p_6, 5), (p_7, 2), (p_8, 0)$.

Dominance Counting

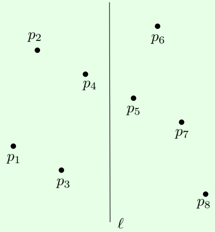
Let P be a set of n points in $\mathbb{N}^2$. Find, for each point $p \in P$, the number of points in P that are dominated by p.

Trivial: $O(n^2)$
We will do in the class: $O(n \log^2 n)$ time.
You will do as an exercise: $O(n \log n)$ time.

Dominance Counting

Divide: Find a vertical line ℓ such that P has $\lceil n/2 \rceil$ points on each side of the line.

Example:



Think: How to find such ℓ in $O(n \log n)$ time? How about $O(n)$ time?

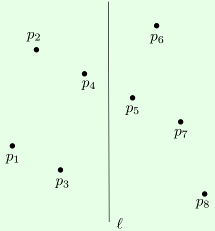
Dominance Counting

Divide:

P_1 = the set of points of P on the left of ℓ

P_2 = the set of points of P on the right of ℓ

Example:



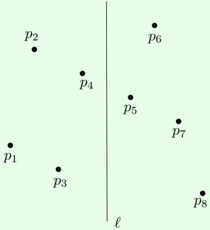
$$P_1 = \{p_1, p_2, p_3, p_4\}$$
$$P_2 = \{p_5, p_6, p_7, p_8\}.$$

Dominance Counting

Divide:

Solve the dominance counting problem on P_1 and P_2 separately.

Example:



On P_1, we have obtained: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2)$.

On P_2, we have obtained: $(p_5, 0), (p_6, 1), (p_7, 0), (p_8, 0)$.

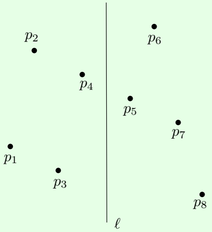
The counts obtained for the points in P_1 are final (**think:** why?).

Dominance Counting

What do we gain?

It remains to count, for each point $p_2 \in P_2$, how many points in P_1 it dominates.

Example:



On P_2, we have obtained: $(p_5, 0), (p_6, 1), (p_7, 0), (p_8, 0)$.

Regarding p_5, for example, we still need to find out that it dominates 2 points from P_1.

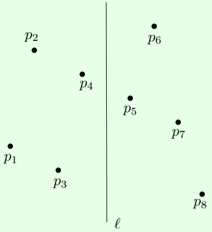
The x-coordinates do not matter any more!

Dominance Counting

What do we gain?

Sort P_1 by **y-coordinate**.
Then, for each point $p_2 \in P_2$, we can obtain the number points in P_1 dominated by p_2 using binary search.

Example:



P_1 in ascending of y-coordinate: p_3, p_1, p_4, p_2.

How to perform binary search to obtain the fact that p_5 dominates 2 points in P_1?

- Search using the y-coordinate of p_5.

Dominance Counting

Analysis:

Let $f(n)$ be the worst-case running time of the algorithm on n points. Then:

$$f(n) \leq 2f(\lceil n/2 \rceil) + O(n \log n)$$

which solves to $f(n) = O(n \log^2 n)$.

Matrix Multiplication

Matrix Multiplication

Problem: Given two $n \times n$ matrices A and B, compute their product AB.

We store an $n \times n$ matrix with an array of length n^2 in "row-major" order.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is stored as $(1, 2, 3, 4)$.

Note that any $A[i, j]$ — the element of A at the i-th row and j-th column — can be accessed in $O(1)$ time.

Trivial: $O(n^3)$ time
We will do in the class: $O(n^{2.81})$ time for n being a power of 2
You will do as an exercise: $O(n^{2.81})$ time for any n.

Matrix Multiplication

Warm Up: Suppose we want to compute $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. How many multiplication operations do we need to perform?

Trivial: 8.

Non-trivial: 7.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} p_5 + p_4 - p_2 + p_6 & p_1 + p_2 \\ p_3 + p_4 & p_1 + p_5 - p_3 - p_7 \end{bmatrix}$$

where

$$\begin{aligned} p_1 &= a(f - h) \\ p_2 &= (a + b)h \\ p_3 &= (c + d)e \\ p_4 &= d(g - e) \\ p_5 &= (a + d)(e + h) \\ p_6 &= (b - d)(g + h) \\ p_7 &= (a - c)(e + f) \end{aligned}$$

Matrix Multiplication (Strassen's Algorithm)

Recall that the input A and B are order-n (i.e., $n \times n$) matrices. Assume for simplicity that n is a power of 2. Divide each of A and B into 4 submatrices of order $n/2$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

It is easy to verify:

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

How many order-$(n/2)$ matrix multiplications do we need?

Trivial: 8.
Non-trivial: 7 — see the next slide.

Matrix Multiplication

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} p_5 + p_4 - p_2 + p_6 & p_1 + p_2 \\ p_3 + p_4 & p_1 + p_5 - p_3 - p_7 \end{bmatrix}$$

$$\begin{aligned} p_1 &= A_{11}(B_{12} - B_{22}) \\ p_2 &= (A_{11} + A_{12})B_{22} \\ p_3 &= (A_{21} + A_{22})B_{11} \\ p_4 &= A_{22}(B_{21} - B_{11}) \\ p_5 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\ p_6 &= (A_{12} - A_{22})(B_{21} + B_{22}) \\ p_7 &= (A_{11} - A_{21})(B_{11} + B_{12}) \end{aligned}$$

If $f(n)$ is the worst-case time of computing the product of two order-n matrices, then each of p_i $(1 \le i \le 7)$ can be computed in $f(n/2) + O(n^2)$ time.

Matrix Multiplication

Therefore:

$$f(n) = 7f(n/2) + O(n^2)$$

which solves to $f(n) = O(n^{\log_2 7}) = O(n^{2.81})$.

Matrix Multiplication

Remark: Matrix multiplication is one of the biggest open problems in computer science. Currently the fastest algorithm runs in $O(n^{2.373})$ time. It is not clear how much more improvement is possible (although many people believe that it could be eventually lowered to $O(n^2)$).