ENGG1410F Tutorial
More on Similarity Transformation

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I: Proving Non-diagonalizability

We will start by seeing an example where we want to prove that the following matrix is not diagonalizable:

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

I: Proving Non-diagonalizability

Goal: Prove that we won't be able to find 3 linearly independent eigenvectors of $\boldsymbol{A}$.

First, find the eigenvalues of $\boldsymbol{A}$: $\lambda_1 = 1$ and $\lambda_2 = 2$.

We will prove:

- The eigenspace of λ_1 has dimension 1
  namely, any two eigenvectors of λ_1 must be linearly dependent.
- The same is true for λ_2.

This will complete the proof.

I: Proving Non-diagonalizability

Let us first focus on $\lambda_1 = 1$. We want to solve the equation:

$$\begin{aligned} (\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} &= 0 \Rightarrow \\ \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \mathbf{0} \end{aligned}$$

We can see that there are two useful equations. In other words, there is only one unconstrained variable. Therefore, *eigenspace*(λ_1) has dimension 1.

I: Proving Non-diagonalizability

Next, focus on $\lambda_2 = 2$. We want to solve the equation:

$$
\begin{aligned}
(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x} &= 0 \Rightarrow \\
\begin{bmatrix} -3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \mathbf{0}
\end{aligned}
$$

We can see that there are two useful equations. In other words, there is only one unconstrained variable. Therefore, $\textit{eigenspace}(\lambda_2)$ has dimension 1.

We now can conclude that $\boldsymbol{A}$ is not diagonalizable.

II: Transitivity of Diagonalizability

Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be three $n \times n$ matrices for some integer n.

> If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ and $\boldsymbol{B}$ is similar to $\boldsymbol{C}$,
> then $\boldsymbol{A}$ is similar to $\boldsymbol{C}$.

This is an exercise in the last week's exercise list.

III: Proof of Similarity

Prove:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

is similar to

$$\boldsymbol{B} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

We will give two ways to do this.

III: Proof of Similarity

Method 1: Use transitivity.

Verify that $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvalues: 3 and 2.

- By the way, if they do not, then immediately they are not similar.

Hence, $\boldsymbol{A}$ can be diagonalized into $\boldsymbol{P}^{-1} diag[3, 2]\boldsymbol{P}$, and
$\boldsymbol{B}$ can be diagonalized into $\boldsymbol{Q}^{-1} diag[3, 2]\boldsymbol{Q}$.

In other words, $\boldsymbol{A}$ and $\boldsymbol{B}$ are both similar to $diag[3, 2]$. Therefore, $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar to each other.

III: Proof of Similarity

Method 2: Finding an explicit form.

We will try to find an invertible matrix $\boldsymbol{P} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $\boldsymbol{A} = \boldsymbol{PBP}^{-1}$. Equivalently, we want to have $\boldsymbol{AP} = \boldsymbol{PB}$, that is:

$$\begin{aligned} \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \\ \begin{bmatrix} x - z & y - w \\ 2x + 4z & 2y + 4w \end{bmatrix} &= \begin{bmatrix} 3x & x + 2y \\ 3z & z + 2w \end{bmatrix} \end{aligned}$$

III: Proof of Similarity

Method 2: Finding an explicit form.

This gives the following equation set:

$$\begin{aligned} x - z &= 3x \\ y - w &= x + 2y \\ 2x + 4z &= 3z \\ 2y + 4w &= z + 2w \end{aligned}$$

You can verify that the set of solutions $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is

$$\left\{ \begin{bmatrix} -u/2 \\ u/2 - v \\ u \\ v \end{bmatrix} \;\middle|\; u \in \mathbb{R}, v \in \mathbb{R} \right\}.$$

III: Proof of Similarity

Method 2: Finding an explicit form.

Let us try $u = 2, v = 0$. This gives $\boldsymbol{P} = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$.

Since $det(\boldsymbol{P}) \neq 0$, we know that $\boldsymbol{P}$ is invertible. We can now conclude that $\boldsymbol{A}$ is similar to $\boldsymbol{B}$.