ENGG1410-F Tutorial 4

Shangqi Lu

Department of Computer Science and Engineering
The Chinese University of Hong Kong

I: Problems 4 and 5 in the Exercise list on "Matrix Inverses"

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(1) Use Gauss-Jordan elimination to calculate the inverse of A.

(2) Use the "inverse formula" to calculate the inverse of A.

II: Problem 1 in the Exercise list on "Matrix Inverses"

Consider the following linear system:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ 3x_1 + x_2 + x_3 + x_4 = a \\ x_2 + 2x_3 + 2x_4 = 3 \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 = a \end{cases}$$

Depending on the value of a, when does the system have no solution, a unique solution, and infinitely many solutions?

Solution

Given a linear system $Ax = b$, and the augmented matrix is $\widetilde{A} = [A|b]$.

Consistency Criterion Theorem. The linear system has:

1. no solution if and only if rank $A <$rank $\widetilde{A}$;
2. exactly one solution if and only if rank $A =$rank $\widetilde{A} = n$;
3. infinitely many solutions if and only if rank $A =$rank $\widetilde{A} < n$.

Solution

Consider the augmented matrix $\widetilde{A}$:

$$\widetilde{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & a \\ 0 & 1 & 2 & 2 & 3 \\ 5 & 4 & 3 & 3 & a \end{array}\right]$$

Solution

Apply elementary row operations to convert the augmented matrix $\widetilde{A}$ into row echelon form:

$$\widetilde{A} \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 & a-3 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & -1 & -2 & -2 & a-5 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & -2 & -2 & -2 & a-3 \\ 0 & -1 & -2 & -2 & a-5 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 2 & 2 & a+3 \\ 0 & 0 & 0 & 0 & a-2 \end{array}\right]$$

Solution

Now we can analyze the solutions of the linear system:

- If $a \neq 2$, then rank $\widetilde{A} = 4$ whereas rank $A = 3$. In this case, the system has no solution.
- If $a = 2$, then rank $A =$ rank $\widetilde{A} = 3$, which is smaller than the number of variables. Hence, the system has infinitely many solutions.
- Regardless of the value of a, the linear system never has a unique solution.

III: Problem 6 in the Exercise list on "Matrix Inverses"

Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 5 & 9 & 1 \end{bmatrix}$$

IV: Problem 7 in the Exercise list on "Matrix Inverses"

Let A be an $n \times n$ matrix. Also, let I be the $n \times n$ identity matrix. Prove: if $A^3 = 0$, then

$$(I - A)^{-1} = I + A + A^2$$

V: Problem 1 in the Exercise list on "Dimensions"

Let V be the set of following 1×4 vectors. Find the dimension of V.

$$\begin{bmatrix} 3 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & 1 & 2 \end{bmatrix}$$

VI: Problem 3 in the Exercise list on "Dimensions"

For each set V of vectors given below, find its dimension and give a basis:

(a) V is the set of 2D points given by $y = x$ (here, we regard each point (x, y) as a 1×2 vector $[x, y]$);

(b) V is the set of 2D points given by $y = x + 1$.

Solution

(a) Dimension: 1. A basis: $\{[1, 1]\}$;

(b) Dimension: 2. A basis: $\{[0, 1], [-1, 0]\}$.

Remark: Here is an intuitive explanation why (b) has one more dimension than (a). To specify a line crossing the origin (such as $y = x$), you only need to give its slope; hence, the dimension of the line is 1. To specify a line that does not pass the origin (such as $y = x + 1$), you need to specify first its scope and then how to shift the line to the position you want; hence, the dimension becomes 2.

VII: Problem 2 in the Exercise list on "Dimensions"

Let V be the set of 1×4 vectors $[2x - 3y, x + 2y, -y, 4x]$ with $x, y \in \mathbb{R}$. Find the dimension of V and gives basis of V.