ENGG1410-F Tutorial 10

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Problem 1.

Let C be the curve from point $p = (0, 0)$ to point $q = (2, 4)$ on the parabola $y = x^2$. Calculate $\int_C (x^2 - y^2)dx$.

Problem 2.

Let $\boldsymbol{r}(t) = [t, t^2, t^3]$ and $\boldsymbol{f}(x, y, z) = [x - y, y - z, z - x]$. Let C be the curve from the point of $t = 0$ to point of $t = 1$. Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$.

Problem 3.

Let $\boldsymbol{r}(t) = [t, t^2, t^3]$ and $\boldsymbol{f}(x, y, z) = [x - y, y - z, z - x]$. Let C be the curve from the point of $t = 1$ to point of $t = 0$. Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$.

Problem 4.

Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, where $\boldsymbol{f}(x, y) = [y^2, -x^2]$, and C is the arc from $(0, 0)$ to $(1, 4)$ on the curve $y = 4x^2$.

Problem 5.

Calculate

$$\int_C xydx + x^2y^2dy$$

where C is the quarter-arc from $(1,0)$ to $(0,1)$ on the circle $x^2 + y^2 = 1$.

Problem 6.

Let $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = cos(t)$ and $y(t) = sin(t)$. Let p be the point given by $t = \pi/4$. Calculate $\frac{dx}{ds}$ at p.

Problem 7.

Let $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$. Let p be the point given by $t = t_0$. Prove that $[\frac{dx}{ds}(t_0), \frac{dy}{ds}(t_0), \frac{dz}{ds}(t_0)]$ is a unit tangent vector at p.

Problem 7 - Solution.

Proof.

$$\frac{dx}{ds} = \frac{dx/dt}{ds/dt} = \frac{dx/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}} \quad (1)$$

Similarly,

$$\frac{dy}{ds} = \frac{dy/dt}{ds/dt} = \frac{dy/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}} \quad (2)$$

$$\frac{dz}{ds} = \frac{dz/dt}{ds/dt} = \frac{dz/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}} \quad (3)$$

Problem 7 - Solution.

From $(1), (2), (3)$, we know that

$$\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right] = \frac{[x'(t), y'(t), z'(t)]}{\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}}$$

which proves that $[\frac{dx}{ds}(t_0), \frac{dy}{ds}(t_0), \frac{dz}{ds}(t_0)]$ is a tangent vector at p. Furthermore,

$$\left|\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]\right|^2 = \frac{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = 1$$

which means that $[\frac{dx}{ds}(t_0), \frac{dy}{ds}(t_0), \frac{dz}{ds}(t_0)]$ is a unit vector.

Problem 8.

This problem allows you to see the equivalence of line integral by arc length and line integral by coordinate. Let $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = cos(t)$ and $y(t) = sin(t)$. Convert $\int_C xdx + \int_C y^2dy$ to line integral by arc length.