ENGG1410-F: Quiz 3

Name: **Student ID:**

Problem 1 (40%). Find an equation of the plane that passes points $A(1,0,0), B(0,2,0)$, and $C(0,0,3)$.

Solution.

$$\overrightarrow{AB} = [-1,2,0]$$
$$\overrightarrow{AC} = [-1,0,3]$$

Hence we can get a normal vector $\boldsymbol{u}$ of the plane as

$$\begin{aligned} \boldsymbol{u} &= \overrightarrow{AB} \times \overrightarrow{AC} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = [6,3,2]. \end{aligned}$$

Let $P = (x,y,z)$ be a point on the plane. Then, the vector $\overrightarrow{AP} = [x-1,y,z]$ must be perpendicular to $\boldsymbol{u}$, which gives:

$$\begin{aligned} \overrightarrow{AP} \cdot \boldsymbol{u} &= 0 \Rightarrow \\ 6(x-1) + 3y + 2z &= 0 \end{aligned}$$

which is an equation of the plane.

Problem 2 (20%). Consider the curve $\boldsymbol{r}(t) = [t,t,t^2]$. Let C be the arc of the curve defined by increasing t from 0 to 1. Calculate

$$\int_C \frac{1}{\sqrt{4t^2+2}}\, ds.$$

Solution. Write $\boldsymbol{r}(t) = [x(t), y(t), z(t)] = [t,t,t^2]$. Therefore:

$$\begin{aligned} \int_C \frac{1}{\sqrt{4t^2+2}}\, ds &= \int_0^1 \frac{1}{\sqrt{4t^2+2}} \frac{ds}{dt}\, dt \\ &= \int_0^1 \frac{1}{\sqrt{4t^2+2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_0^1 \frac{1}{\sqrt{4t^2+2}} \sqrt{1+1+4t^2}\, dt \\ &= \int_0^1 dt = 1 \end{aligned}$$

Problem 3 (40%). Consider the scalar function $f(x, y, z) = x^2y + z^2$. Find the maximum rate of change at point $P = (1, 1, 1)$.

(Note: the maximum rate of change is the largest directional derivative at P.)

Solution. First compute the gradient of f:

$$\nabla f \quad = \quad \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [2xy, x^2, 2z]$$

The maximum rate of change is the directional derivative of the unit vector $\boldsymbol{u}$ that has the same direction as ∇f.

Now, calculate ∇f at point P:

$$\nabla f(P) \quad = \quad [2, 1, 2].$$

Hence:

$$\boldsymbol{u} \quad = \quad \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]$$

The directional derivative at P in the direction of $\boldsymbol{u}$ is

$$\begin{aligned} \nabla f(P) \cdot \boldsymbol{u} \quad &= \quad [2, 1, 2] \cdot \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right] \\ &= \quad 3. \end{aligned}$$