Lecture Notes: Dimension, Span, and Linear Transformation

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1 Dimension of a Set of Vectors

Let V be a possibly infinite set of vectors. These vectors are either all row vectors or all column vectors of the same length. We define the concepts *dimension* and *basis* for V as follows:

Definition 1. *The* **dimension** *of V is the maximum number d of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_d$ we can find in V such that $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_d$ are linearly independent. The set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_d\}$ is a* **basis** *of V.*

Example 1. Let V be the set of all possible 1×2 vectors. V has dimension 2. The set of vectors $[1, 0], [0, 1]$ is a basis of V. Note that bases are not unique: e.g., $[1, 0], [0, 2]$ form another basis. □

Example 2. Let V be the set of all possible 1×2 vectors $[x, y]$ satisfying $y = 3x$. V has dimension 1. A basis is $[1, 3]$. You can verify that any two vectors in V must be linearly dependent. □

Immediately, we have:

Lemma 1. *Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_d\}$ be a basis of V. Any vector $\boldsymbol{u} \in V$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_d$.*

The proof should have become trivial for you at this moment. You are encouraged to verify the lemma on the V in Examples 1 and 2.

When V is finite, its dimension and basis can be conveniently understood by resorting to a matrix. For example, suppose that V has m $1 \times n$ vectors. Define an $m \times n$ matrix $\boldsymbol{M}$ whose i-th row is the i-th vector of V, for each $1 \le i \le m$. Then:

- The dimension d of V is simply the rank of $\boldsymbol{M}$.
- A basis of V can be any set of d rows in $\boldsymbol{M}$ which are linearly independent.

2 Span

Let B be a possibly infinite set of vectors. These vectors are either all row vectors or all column vectors of the same length. We define the *span* of B as follows:

Definition 2. *The* **span** *of B is the set of vectors that can be obtained as linear combinations of the vectors in B.*

Note that the span V of B has an infinite size, and that $B \subseteq V$. V is sometimes also referred to as the *vector space* determined by B.

Example 3. Let $B = \{[1,0],[0,1]\}$; the span of B is the set of all possible 1×2 vectors. As another example, let $B = \{[1,0],[0,1],[2,3]\}$; the span of B is still the set of all possible 1×2 vectors. $\square$

Example 4. Let $B = \{[1,0,0],[0,1,0]\}$; the span of B is the set of all possible 1×3 vectors $[x,y,z]$ satisfying $z = 0$. As another example, Let $B = \{[1,0,0],[0,1,0],[2,3,0]\}$; the span of B still the same. But if $B = \{[1,0,0],[0,1,0],[0,0,1]\}$, then the span of B becomes the set of all possible 1×3 vectors. $\square$

Lemma 2. *Let V be the span of B. The dimension of V equals the dimension of B.*

Proof. Let d_V be the dimension of V, and d_B be the dimension of B. To establish the lemma, we need to prove two directions:

Direction 1: $d_V \ge d_B$. Suppose on the contrary that $d_V < d_B$. Then, any set of at least $d_V + 1$ vectors in V must be linearly dependent. As $B \subseteq V$, it follows that any set of $d_B \ge d_V + 1$ vectors in B must be linearly dependent. But this contradicts the fact that the dimension of B is d_B.

Direction 2: $d_B \ge d_V$. Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_{d_B}\}$ be a basis of B. By definition of span, we know that any vector in V is a linear combination of $\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_{d_B}$. Hence, $d_B \ge d_V$, by definition of d_V. $\square$

You are encouraged to verify the lemma on the B in Examples 3 and 4. Next we give a slightly more sophisticated example with an infinite B.

Example 5. Consider that B is the set of vectors $[x,y]$ satisfying

$$0 \le x \le 1$$
$$0 \le y \le 1$$

The dimension of B is 2. What is the span of B? The answer is the set V of all possible 1×2 vectors. The dimension of V is 2, too. $\square$

3 Linear Transformation

Let V_1 be a set of $n \times 1$ vectors. Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then, given a vector $\boldsymbol{v} \in V_1$, define function

$$\boldsymbol{f}(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}.$$

Note that $\boldsymbol{f}(\boldsymbol{v})$ is an $m \times 1$ vector. Define:

$$V_2 = \{\boldsymbol{f}(\boldsymbol{v}) \mid \boldsymbol{v} \in V_1\} \tag{1}$$

We say that function $\boldsymbol{f}$ is a *linear transformation* from V_1 to V_2. Also, we refer to $\boldsymbol{f}(\boldsymbol{v})$ as the *image* of $\boldsymbol{v}$.

Example 6. Let V_1 be all the 2×1 vectors $\begin{bmatrix} x \\ y \end{bmatrix}$. Define $\boldsymbol{f}(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ where

$$u = 2x + y$$
$$v = -x - y$$
$$w = 3x + 4y$$

The linear transformation can also be written as

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

□

Lemma 3. *The dimension of V_2 is at most the dimension of V_1.*

Proof. Let d be the dimension of V_1, and $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_d\}$ be a basis of V_1. We will show that any vector $\boldsymbol{u} \in V_2$ is a linear combination of $\boldsymbol{f}(\boldsymbol{v}_1), ..., \boldsymbol{f}(\boldsymbol{v}_d)$. This will complete the proof.

Without loss of generate, suppose that $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}$ for some $\boldsymbol{v} \in V_1$. If $\boldsymbol{v} = \boldsymbol{v}_i$ for some $1 \le i \le d$, then

$$\boldsymbol{u} = 1 \cdot \boldsymbol{A}\boldsymbol{v}_i + 0 \cdot \sum_{j \neq i} \boldsymbol{A}\boldsymbol{v}_j = 1 \cdot \boldsymbol{f}(\boldsymbol{v}_i) + 0 \cdot \sum_{j \neq i} \boldsymbol{f}(\boldsymbol{v}_j);$$

and our claim is true.

Now consider that $\boldsymbol{v} \notin \{\boldsymbol{v}_1, ..., \boldsymbol{v}_d\}$. We know that $\boldsymbol{v}$ must be a linear combination of $\boldsymbol{v}_1, ..., \boldsymbol{v}_d$:

$$\boldsymbol{v} = \sum_{i=1}^{d} c_i \cdot \boldsymbol{v}_i$$

for some real-valued constants $c_1, ..., c_d$. Thus:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{v} &= \sum_{i=1}^{d} c_i \cdot \boldsymbol{A}\boldsymbol{v}_i \\ \Rightarrow \boldsymbol{u} &= \sum_{i=1}^{d} c_i \cdot \boldsymbol{f}(\boldsymbol{v}_i) \end{aligned}$$

□

The lemma confirms the following intuition: no new information is generated by the linear transformation. To understand this, consider Example 6 again. V_1 clearly has dimension 2. The set V_2 obtained by $\boldsymbol{f}$ contains 3×1 vectors. So it may appear that V_2 *had* a dimension of 3. The above lemma shows that this is impossible: indeed, the dimension of V_2 is 2.