Exercises: Tangent and Gradient

Problem 1. Let $\boldsymbol{f}(t) = [3,4]+t[1,2]$. Give a tangent vector of the curve at the point corresponding to $\boldsymbol{f}(2)$.

Solution. We know that $\boldsymbol{f}(t) = [3+t, 4+2t]$. Taking the derivative gives $\boldsymbol{f}'(t) = [1,2]$. Hence, $[1,2]$ is a tangent vector at the point correspond to $\boldsymbol{f}(2)$.

Problem 2. Let $\boldsymbol{f}(t) = [\sin(t), \cos(t^3), 5t^2]$. Give a tangent vector of the curve at the point corresponding to $\boldsymbol{f}(2)$.

Solution. Since $\boldsymbol{f}'(t) = [\cos(t), -3t^2\sin(t^3), 10t]$, a tangent vector at the point corresponding to $\boldsymbol{f}(2)$ is $\boldsymbol{f}'(2) = [\cos(2), -12\sin(8), 20]$.

Problem 3. Give a tangent vector of point $(2, \sqrt{2})$ on the ellipse $x^2 + \frac{y^2}{2} = 5$.

Solution. Introduce $x(t) = \sqrt{5}\cos(t)$ and $y(t) = \sqrt{10}\sin(t)$. Hence, the curve can be described by $\boldsymbol{f}(t) = [x(t), y(t)]$. We thus have: $\boldsymbol{f}'(t) = [-\sqrt{5}\sin(t), \sqrt{10}\cos(t)]$. Point $(2, \sqrt{2})$ corresponds to $\boldsymbol{f}(t_0)$ with $\sqrt{5}\cos(t_0) = 2$ and $\sqrt{10}\sin(t_0) = \sqrt{2}$. Hence, a tangent vector at the point is $\boldsymbol{f}'(t_0) = [-\sqrt{5}\sin(t_0), \sqrt{10}\cos(t_0)] = [-1, 2\sqrt{2}]$.

Problem 4. Let $\boldsymbol{f}(t) = [t^2, -2t, -t^3]$. Give a tangent vector of the curve at point $(9, -6, -27)$.

Solution. First, we get $\boldsymbol{f}'(t) = [2t, -2, -3t^2]$. Note that point $(9,-6,-27)$ corresponds to $\boldsymbol{f}(3)$. Hence, a tangent vector at the point is $\boldsymbol{f}'(3) = [6, -2, -27]$.

Problem 5. Compute the following gradients:

1. $\nabla f(3,4)$ where $f(x,y) = (4x+3)(2y-1)$.
2. $\nabla f(3,4,5)$ where $f(x,y,z) = 3x^2yz$.

Solution.

- $\nabla f(x,y) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}] = [4(2y-1), 2(4x+3)]$. Hence, $\nabla f(x,y) = [28, 30]$.
- $\nabla f(x,y,z) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = [6xyz, 3x^2z, 3x^2y]$. Hence, $\nabla f(3,4,5) = [360, 135, 108]$.

Problem 6. Let $g(x,y) = (f(x,y))^c$. Prove that $\nabla g(x,y) = c(f(x,y))^{c-1}\nabla f(x,y)$.

Proof. $\nabla g(x,y) = [\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}] = [c(f(x,y))^{c-1}\frac{\partial f}{\partial x}, c(f(x,y))^{c-1}\frac{\partial f}{\partial y}] = c(f(x,y))^{c-1}[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}] = c(f(x,y))^{c-1}\nabla f(x,y)$. □

Problem 7. Let $f(x,y,z) = 3x^2yz$. Let $\boldsymbol{u} = [1/3, 1/3, 1/3]$. Compute directional derivative of $f(x,y,z)$ in the direction of $\boldsymbol{u}$ at point $(5,2,3)$.

Solution. $\nabla f(x,y,z) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = [6xyz, 3x^2z, 3x^2y]$. Let us normalize $\boldsymbol{u}$ into $\boldsymbol{v} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{[1/3,1/3,1/3]}{\sqrt{3}/3} = [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]$. Hence, the directional derivative of $f(5,2,3)$ towards the direction of $\boldsymbol{v}$ (namely, of $\boldsymbol{u}$) is $\nabla f(5,2,3) \cdot \boldsymbol{v} = [180, 225, 150] \cdot [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}] = 555/\sqrt{3}$.

Problem 8. Let $f(x,y,z) = 3x^2yz$. Find the unit vector $\boldsymbol{u}$ that maximizes the directional derivative of $f(x,y,z)$ in the direction of $\boldsymbol{u}$ at point $(5,2,3)$.

Solution. As explained earlier, $\nabla f(5,2,3) = [180, 225, 150]$. Hence, the directional derivative of $f(5,2,3)$ is maximized in direction of the unit vector $\boldsymbol{u} = \frac{[180,225,150]}{|[180,225,150]|} = \frac{[180,225,150]}{\sqrt{4221}}$.