Exercises: Dot Product and Cross Product

Problem 1. For the following directed segments, give the vectors they define:

1. $\overrightarrow{(1,2),(2,3)}$
2. $\overrightarrow{(10,20),(11,21)}$
3. $\overrightarrow{(1,-2),(2,3)}$
4. $\overrightarrow{(1,-2,0),(2,3,10)}$

Solution:

1. $[1, 1]$.
2. $[1, 1]$
3. $[1, 5]$
4. $[1, 5, 10]$

Problem 2. In each of the following cases, indicate whether $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction (i.e., whether their angle is 0):

1. $\boldsymbol{a} = [1, 1], \boldsymbol{b} = [2, 2]$
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [20, 40, 60]$
3. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [2, -4, 6]$

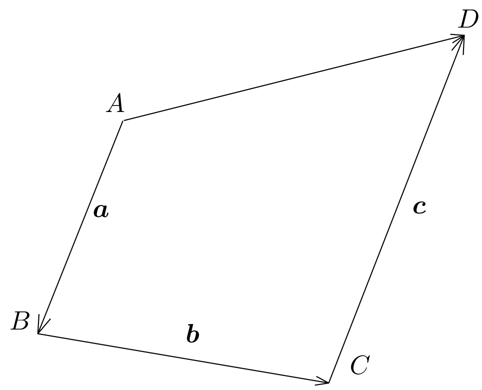
Solution:

1. Yes
2. Yes
3. No

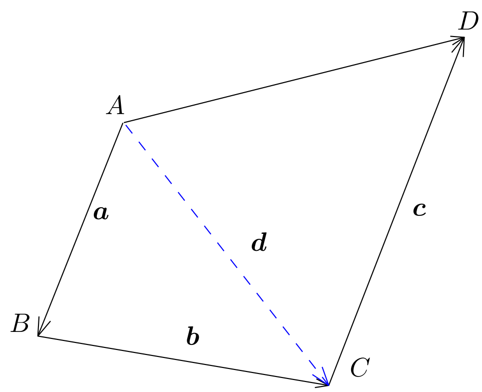
Problem 3. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be 2d vectors such that $\boldsymbol{a} + \boldsymbol{b} = [3, 5]$, and $\boldsymbol{a} - \boldsymbol{b} = [4, 6]$. What are $\boldsymbol{a}$ and $\boldsymbol{b}$?

Solution: Since $(\boldsymbol{a} + \boldsymbol{b}) + (\boldsymbol{a} - \boldsymbol{b}) = 2\boldsymbol{a} = [3, 5] + [4, 6] = [7, 11]$, we know $\boldsymbol{a} = [3.5, 5.5]$. From this we get $\boldsymbol{b} = [-0.5, 0.5]$.

Problem 4. Let A, B, C, D be 4 points in $\mathbb{R}^d$. Suppose that directed segments $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CD}$ define vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively; see the figure below. Prove that $\overrightarrow{AD}$ is an instantiation of $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$.



Solution: The directed segment $\overrightarrow{AC}$ defines vector $\boldsymbol{d} = \boldsymbol{a}+\boldsymbol{b}$. Hence, $\overrightarrow{AD}$ defines $\boldsymbol{d}+\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$.



Problem 5. Give the result of $\boldsymbol{a} \times \boldsymbol{b}$ for each of the following:

1. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$.
2. $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = [3, 2, 1]$.

Solution:

1. $\boldsymbol{a} \times \boldsymbol{b} = \left[\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}\right] = [-4, 8, -4]$.
2. $\boldsymbol{a} = [1, -1, 1]$. Then it is easy to obtain that $\boldsymbol{a} \times \boldsymbol{b} = [-3, 2, 5]$.

Problem 6. In each of the following, you are given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Give the value of $\cos\gamma$, where γ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

1. $\boldsymbol{a} = [1, 2], \boldsymbol{b} = [2, 5]$
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$

Solution:

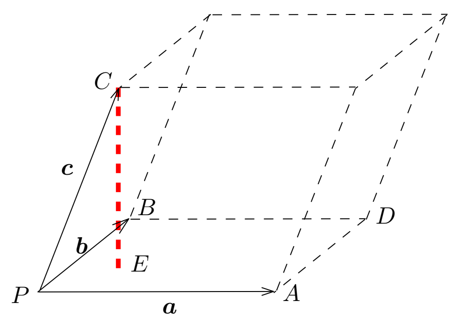
1. $\cos\gamma = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \frac{12}{\sqrt{5}\cdot\sqrt{29}} = \frac{12}{\sqrt{145}}$.

2. $\frac{5}{7}$.

Problem 7. This exercise explores the usage of dot product for calculation of projection lengths. Consider points $P(1,2,3), A(2,-1,4), B(3,2,5)$. Let ℓ be the line passing P and A. Now, let us project point B onto ℓ; denote by C the projection. Calculate the distance between P and C.

Solution: Let γ be the angle between vectors $\overrightarrow{PA}$ and $\overrightarrow{PB}$. We have $|\overrightarrow{PC}| = |\overrightarrow{PB}||\cos\gamma| = |\overrightarrow{PB}|\frac{\overrightarrow{PA}\cdot\overrightarrow{PB}}{|\overrightarrow{PA}||\overrightarrow{PB}|} = \frac{\overrightarrow{PA}\cdot\overrightarrow{PB}}{|\overrightarrow{PA}|}$. Given $\overrightarrow{PA} = [1,-3,1]$ and $\overrightarrow{PB} = [2,0,2]$, we know that which equals $\frac{\overrightarrow{PA}\cdot\overrightarrow{PB}}{|\overrightarrow{PA}|} = \frac{4}{\sqrt{11}}$.

Problem 8. Let $\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ be directed segments that are not in the same plane. They determine a parallelepiped as shown below:



Suppose that $\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ define vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. Prove that the volume of the parallelepiped equals $|(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}|$.

Proof: Let E be the projection of point C onto the plane defined by P, A, B (see the above figure). Denote by $\overline{CE}$ the segment connecting C and E, and by $\overline{CE}$ its length. Clearly, the volume of the parallelepiped equals $area(PADB)\cdot|\overline{CE}|$. From the notes of Lecture 2, we know that $|\boldsymbol{a}\times\boldsymbol{b}|$ is exactly $area(PADB)$. So to complete the proof, we need to show:

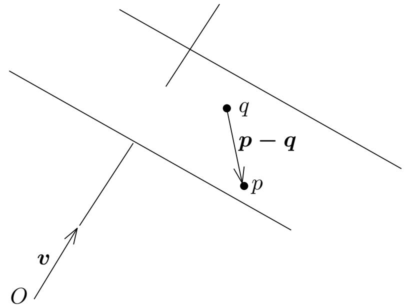
$$\begin{aligned} |(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}| &= |\boldsymbol{a}\times\boldsymbol{b}||\overline{CE}| \quad\Leftrightarrow \\ |\boldsymbol{a}\times\boldsymbol{b}||\boldsymbol{c}||\cos\gamma| &= |\boldsymbol{a}\times\boldsymbol{b}||\overline{CE}| \end{aligned} \tag{1}$$

where γ is the angle between the directions of $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{c}$. To prove Equation 1, it suffices to prove

$$|\boldsymbol{c}||\cos\gamma| = |\overline{CE}|$$

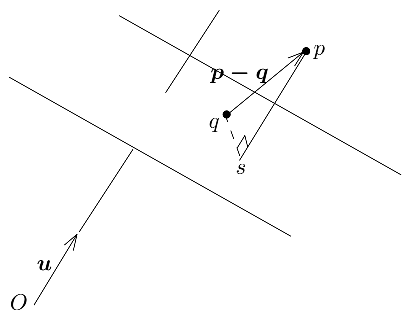
which is true because γ is also the angle between $\overrightarrow{PC}$ and $\overline{CE}$.

Problem 9. Given a point $p(x,y,z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x,y,z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{v}$ be a non-zero 3d vector. Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{v}$. Prove that for any p on ρ, it holds that $(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{v} = 0$.



Proof: The equation obviously holds if $q = p$. Now consider the case where $q \neq p$, as shown in the above figure. We know that the directions of $\boldsymbol{v}$ and $\boldsymbol{p} - \boldsymbol{q}$ are orthogonal. Therefore, $(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{v} = 0$. □

Problem 10. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{u}$ be a unit 3d vector (i.e., $|\boldsymbol{u}| = 1$). Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{u}$. Prove that for any p in $\mathbb{R}^3$, its distance to ρ equals $|(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{u}|$.



Proof: If p falls on ρ, then the equation follows from the result of Problem 6. Otherwise, let s be the projection of p onto ρ. See the above figure. Let γ be the angle between the two segments $\overline{pq}$ and $\overline{ps}$. Hence:

$$|ps| = |pq||\cos\gamma|$$

It suffices to prove that

$$\begin{aligned} |pq||\cos\gamma| &= |(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{u}| \\ &= |(\boldsymbol{p} - \boldsymbol{q})||\boldsymbol{u}||\cos\theta| \end{aligned}$$

where θ is the angle between the directions of $\boldsymbol{u}$ and $\boldsymbol{p} - \boldsymbol{q}$. The above is true because (i) $|pq| = |(\boldsymbol{p} - \boldsymbol{q})|$ and (ii) either $\theta = \gamma$ or $\theta = 180° - \gamma$. We thus complete the proof. □

Problem 11. Consider the plane $x + 2y + 3z = 4$ in $\mathbb{R}^3$. Calculate the distance from point $(0, 0, 0)$ to the plane.

Solution: We can re-write the plane's equation as

$$1 \cdot (x - 0) + 2 \cdot (y - 0) + 3 \cdot (z - 4/3) = 0.$$

Hence, $q(0,0,4/3)$ is a point on the plane. Also, we know that the direction of $\boldsymbol{v} = [1,2,3]$ is perpendicular to the plane. Let $\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = [\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}]$. Note that the direction of $\boldsymbol{u}$ is also perpendicular to the plane, and that $|\boldsymbol{u}| = 1$. Therefore, we can now apply the result of the previous problem to compute the distance from $p(0,0,0)$ to the plane as:

$$\left|([0,0,0] - [0,0,4/3]) \cdot [\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}]\right| \quad = \quad \left|-\frac{4}{3} \cdot \frac{3}{\sqrt{14}}\right| = \frac{4}{\sqrt{14}}$$