Exercises: Eigenvalues and Eigenvectors

Problem 1. Find all the eigenvalues and eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Problem 2. Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Prove: $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have exactly the same eigenvalues.

Problem 3 (Hard). Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Prove: $\boldsymbol{A}^{-1}$ exists if and only if 0 is not an eigenvalue of $\boldsymbol{A}$.

Problem 4. Let $\boldsymbol{A}$ be an $n \times n$ square matrix such that $\boldsymbol{A}^{-1}$ exists. Prove: if λ is an eigenvalue of $\boldsymbol{A}$, then $1/\lambda$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

Problem 5. Prove: if $\boldsymbol{A}^2 = \boldsymbol{I}$, then the eigenvalues of $\boldsymbol{A}$ must be 1 or -1.

Problem 6. Suppose that λ_1 and λ_2 are two distinct eigenvalues of matrix $\boldsymbol{A}$. Furthermore, suppose that $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ under λ_1, and that $\boldsymbol{x}_2$ is an eigenvector of $\boldsymbol{A}$ under λ_2. Prove: there does not exist any real number c such that $c\boldsymbol{x}_1 = \boldsymbol{x}_2$.

Problem 7. Suppose that λ_1 and λ_2 are two distinct eigenvalues of matrix $\boldsymbol{A}$. Furthermore, suppose that $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ under λ_1, and that $\boldsymbol{x}_2$ is an eigenvector of $\boldsymbol{A}$ under λ_2. Prove: $\boldsymbol{x}_1 + \boldsymbol{x}_2$ is *not* an eigenvector of $\boldsymbol{A}$.