Exercises: Path Independence

For Problems 1-4, first decide whether the line integral is path independent. If so, calculate the integral on a piecewise smooth arc from point $(0,0)$ to point $(1,1)$ in 2d, or from point $(0,0,0)$ to point $(1,1,1)$ in 3d.

Problem 1. $\int_C 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy)$.

Solution: Let $f_1(x,y) = 2e^{x^2}\cdot x\cos(2y)$ and $f_2(x,y) = -2e^{x^2}\cdot\sin(2y)$. Thus, $\frac{\partial f_1}{\partial y} = -4xe^{x^2}\sin(2y)$ and $\frac{\partial f_2}{\partial x} = -4xe^{x^2}\sin(2y)$. Hence, the integral is path independent.

If you can observe that $g(x,y) = e^{x^2}\cos(2y)$ satisfies $\frac{\partial g}{\partial x} = f_1$ and $\frac{\partial g}{\partial y} = f_2$, the value of the integral can be computed directly as $g(1,1) - g(0,0) = e\cos(2) - 1$.

If you cannot, then evaluate the integral on an easy curve C. For example, let C be the concatenation of two curves: C_1 from $(0,0)$ to $(1,0)$, and C_2 from $(1,0)$ to $(1,1)$. We have

$$\begin{aligned}
\int_{C_1} 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy) &= \int_{C_1} 2e^{x^2}x\cos(2y)\,dx \\
&= \int_0^1 2e^{x^2}x\cos(2\cdot 0)\,dx \\
&= \int_0^1 2e^{x^2}x\,dx \\
&= \int_0^1 e^{x^2}\,d(x^2) = e - 1
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{C_2} 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy) &= -\int_{C_2} 2e^{x^2}\sin(2y)\,dy \\
&= -\int_0^1 2e\sin(2y)\,dy = e\cos(2) - e
\end{aligned}$$

Hence, $\int_C 2e^{x^2}(x\cos(2y)\,dx - \sin(2y)\,dy)$ equals $e - 1 + e\cos(2) - e = e\cos(2) - 1$.

Problem 2. $\int_C (x^2y\,dx - 4xy^2\,dy + 8z^2x\,dz)$.

Solutions: Let $f_1 = x^2y$, $f_2 = -4xy^2$, and $f_3 = 8z^2x$. Hence, $\frac{\partial f_1}{\partial y} = x^2$ and $\frac{\partial f_2}{\partial x} = -4y^2$. Since $\frac{\partial f_1}{\partial y} \neq \frac{\partial f_2}{\partial x}$, we conclude that the integral is not path independent.

Problem 3. $\int_C (e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz)$.

Solutions: Let $f_1 = e^y$, $f_2 = xe^y - e^z$, and $f_3 = -ye^z$. Thus, $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = e^y$, $\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} = 0$, and $\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} = -e^z$. Hence, the integral is path independent.

If you can observe that $g(x,y,z) = xe^y - ye^z$ satisfies $\frac{\partial g}{\partial x} = f_1$, $\frac{\partial g}{\partial y} = f_2$, and $\frac{\partial g}{\partial z} = f_3$, the value of the integral can be computed directly as $g(1,1,1) - g(0,0,0) = 0$.

If you cannot, then evaluate the integral on an easy curve C. For example, let C be the concatenation of three curves: C_1 from $(0,0,0)$ to $(0,0,1)$, C_2 from $(0,0,1)$ to $(0,1,1)$, and C_3 from

$(0,1,1)$ to $(1,1,1)$. We have

$$\begin{aligned}\int_{C_1}(e^y\,dx+(xe^y-e^z)\,dy-ye^z\,dz) &= -\int_{C_1} ye^z\,dz \\ &= -\int_0^1 0e^z\,dz=0.\end{aligned}$$

Also

$$\begin{aligned}\int_{C_2}(e^y\,dx+(xe^y-e^z)\,dy-ye^z\,dz) &= \int_{C_2}(xe^y-e^z)\,dy \\ &= \int_0^1 -e\,dy=-e.\end{aligned}$$

Finally

$$\begin{aligned}\int_{C_3}(e^y\,dx+(xe^y-e^z)\,dy-ye^z\,dz) &= \int_{C_3} e^y\,dx \\ &= \int_0^1 e\,dx=e.\end{aligned} \tag{1}$$

Hence, $\int_C(e^y\,dx+(xe^y-e^z)\,dy-ye^z\,dz)=0-e+e=0$.

Problem 4. $\int_C(4y\,dx+(4x+z)\,dy+(y-2z)\,dz)$.

Solutions: Let $f_1=4y$, $f_2=4x+z$, and $f_3=y-2z$. Thus, $\frac{\partial f_1}{\partial y}=\frac{\partial f_2}{\partial x}=4$, $\frac{\partial f_1}{\partial z}=\frac{\partial f_3}{\partial x}=0$, and $\frac{\partial f_2}{\partial z}=\frac{\partial f_3}{\partial y}=1$. Hence, the integral is path independent.

If you can observe that $g(x,y,z)=4xy+yz-z^2$ satisfies $\frac{\partial g}{\partial x}=f_1$, $\frac{\partial g}{\partial y}=f_2$, and $\frac{\partial g}{\partial z}=f_3$, the value of the integral can be computed directly as $g(1,1,1)-g(0,0,0)=4$.

If you cannot, then evaluate the integral on an easy curve C. For example, let C be the line segment given by $\boldsymbol{r}(t)=[x(t),y(t),z(t)]$ with $x(t)=y(t)=z(t)=t$, and $t\in[0,1]$. Then

$$\begin{aligned}\int_C(4y\,dx+(4x+z)\,dy+(y-2z)\,dz) &= \int_0^1(4t\,\frac{dx}{dt}+(4t+t)\,\frac{dy}{dt}+(t-2t)\,\frac{dz}{dt})dt \\ &= \int_0^1(4t+5t-t)\,dt=4.\end{aligned}$$

Solve Problems 5-8 by resorting to path independence.

Problem 5. Calculate $\int_C d\boldsymbol{r}=\int_C dx+\int_C dy$ where C is a smooth curve from point $p=(1,2)$ to $q=(3,4)$.

Solution: Introduce $g(x,y)=x+y$. Clearly, $\frac{\partial g}{\partial x}=1$ and $\frac{\partial g}{\partial y}=1$. Hence, $\int_C dx+\int_C dy=g(3,4)-g(1,2)=4$.

Problem 6. Calculate $\int_C 2xy\,dx+\int_C x^2\,dy$ where C is a smooth curve from point $p=(1,2)$ to $q=(3,4)$.

Solution: Introduce $g(x,y) = x^2y$. Clearly, $\frac{\partial g}{\partial x} = 2xy$ and $\frac{\partial g}{\partial y} = x^2$. Hence, $\int_C 2xy\,dx + \int_C x^2\,dy = g(3,4) - g(1,2) = 34$.

Problem 7. Calculate $\int_C yz\,dx + \int_C xz\,dy + \int_C xy\,dz$ where C is a smooth curve from point $p = (1,2,3)$ to $q = (3,4,5)$.

Solution: Introduce $g(x,y,z) = xyz$. Clearly, $\frac{\partial g}{\partial x} = yz$, $\frac{\partial g}{\partial y} = xz$, and $\frac{\partial g}{\partial z} = xy$. Hence, $\int_C yz\,dx + \int_C xz\,dy + \int_C xy\,dz = g(3,4,5) - g(1,2,3) = 54$.

Problem 8. Calculate $\int_C yz\,dx + \int_C xz\,dy + \int_C xy\,dz$ where C is the curve given by $\boldsymbol{r}(t) = [\cos(t), \sin(t), 1]$ with $t \in [0, 2\pi]$.

Solution: We already know that $\int_C yz\,dx + \int_C xz\,dy + \int_C xy\,dz$ is path independent. Also observe that C is a closed curve (because $\boldsymbol{r}(0) = \boldsymbol{r}(2\pi)$). In this case, it must hold that $\int_C yz\,dx + \int_C xz\,dy + \int_C xy\,dz = 0$.