ENGG1410-F Tutorial 8

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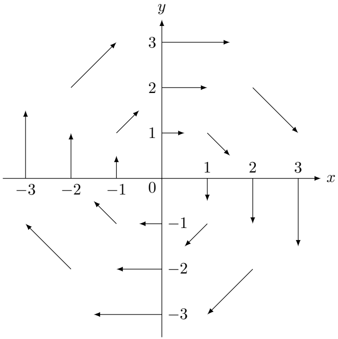
An Example of Vector Field

Recall that a vector function $\boldsymbol{f}$ which takes d real values as its input is also referred as a vector field in $\mathbb{R}^d$.

Consider the vector function $\boldsymbol{f}(x, y) = [y/2, -x/2]$. It can be "visualized" in $\mathbb{R}^2$ as shown in the figure below:



For example, the line segment starting from the point $(-1, 1)$ represents the vector $[1/2, 1/2]$.

Problem 1. Cross Product

Calculate $\boldsymbol{a} \times \boldsymbol{b}$ for each of the following:

1. $\boldsymbol{a} = [3, -3, 1]$, $\boldsymbol{b} = [4, 9, 2]$.
2. $\boldsymbol{a} = [3, -3, 1]$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

Solution

1.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} -3 & 1 \\ 9 & 2 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 3 & -3 \\ 4 & 9 \end{vmatrix}$$

$$= -15\boldsymbol{i} - 2\boldsymbol{j} + 39\boldsymbol{k} = [-15, -2, 39]$$

2. $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} = [1, 1, -1]$, hence

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -3 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix}$$

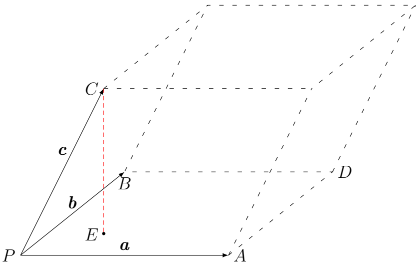
$$= 2\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k} = [2, 4, 6]$$

Problem 2. Dot Product

Consider points $P(1, 2, 3)$, $A(2, -1, 4)$, $B(3, 2, 5)$. Let ℓ be the line passing P and A, denote by C the projection of B onto ℓ. Calculate the distance between P and C.

Problem 3. Cross Product

Consider a parallelepiped as shown below:



Suppose that $\overrightarrow{PA}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$ define vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, respectively. Prove: the volume of the parallelepiped equals $|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$.

Solution

The formal proof of this problem can be found in the solutions to the exercise list on our course homepage. Here we want to point out something more interesting.

Recall that the volume of the parallelepiped also equals to the absolute value of the determinate of the matrix $\boldsymbol{A}$ formed by taking $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as row vectors.

We now verify that $|det(\boldsymbol{A})| = |(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$.

Solution-cont.

Let $\boldsymbol{a} = [a_1, a_2, a_3]$, $\boldsymbol{b} = [b_1, b_2, b_3]$ and $\boldsymbol{c} = [c_1, c_2, c_3]$. We then have

$$det(\boldsymbol{A}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad (1)$$

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot [c_1, c_2, c_3] \\ &= \left(\boldsymbol{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot [c_1, c_2, c_3] \quad (2) \\ &= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{aligned}$$

Comparing (1) and (2) gives $|det(\boldsymbol{A})| = |(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$.

Problem 4. Dot Product

Consider a point $P(1, 2, 3)$ and a vector $\boldsymbol{v} = [1, 1, 1]$ in $\mathbb{R}^3$. Denote by ρ the plane passing P and is perpendicular to the direction of $\boldsymbol{v}$. Give the general expression of ρ.

Solution

Consider an arbitrary point $Q(x, y, z)$ on ρ, we know that the directions of $\overrightarrow{PQ}$ (which defines a vector $\boldsymbol{u}$) and $\boldsymbol{v}$ are orthogonal. Therefore,

$$
\begin{aligned}
& \boldsymbol{u} \cdot \boldsymbol{v} = 0 \\
\Rightarrow \quad & [x-1, y-2, z-3] \cdot [1, 1, 1] = 0 \\
\Rightarrow \quad & x - 1 + y - 2 + z - 3 = 0 \\
\Rightarrow \quad & x + y + z - 6 = 0
\end{aligned}
$$

which gives the plane equation.

Problem 5. Dot Product & Cross Product

Consider points $A(1, 1, 1)$, $B(-1, 1, 0)$ and $C(2, 0, 3)$. Give the general expression of the plane ρ determined by these three points.

Solution

$\overrightarrow{AB}$ defines a vector $\boldsymbol{a} = [-2, 0, -1]$, $\overrightarrow{AC}$ defines a vector $\boldsymbol{b} = [1, -1, 2]$. Consider an arbitrary point $P(x, y, z)$ on ρ, then $\overrightarrow{AP}$ defines a vector $[x-1, y-1, z-1]$ on ρ.

We know that $\boldsymbol{a} \times \boldsymbol{b}$ gives a normal of ρ. Hence,

$$
\begin{aligned}
& (\boldsymbol{a} \times \boldsymbol{b}) \cdot [x-1, y-1, z-1] = 0 \\
\Rightarrow \quad & \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 0 & -1 \\ 1 & -1 & 2 \end{vmatrix} \cdot [x-1, y-1, z-1] = 0 \\
\Rightarrow \quad & x - 3y - 2z + 4 = 0
\end{aligned}
$$

which gives the expression of the plane.

Problem 6. Directional Derivative

Let $f(x, y, z) = x^3 - xy^2 - z$ and $\boldsymbol{u} = [2, -3, 6]$. Compute directional derivative of $f(x, y, z)$ in the direction of $\boldsymbol{u}$ at point $(1, 1, 0)$.

Solution

$\nabla f(x,y,z) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] = [3x^2 - y^2, -2xy, -1]$.

Normalize $\boldsymbol{u}$ into $\boldsymbol{v} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{1}{7}[2, -3, 6]$.

Hence the directional derivative of $f(x,y,z)$ in the direction of $\boldsymbol{v}$ (namely, of $\boldsymbol{u}$) at point $(1,1,0)$ is

$$\nabla f(1,1,0) \cdot \boldsymbol{v} = [2, -2, -1] \cdot \frac{1}{7}[2, -3, 6] = \frac{4}{7}$$

Problem 7. Directional Derivative

Let $f(x, y, z) = x^3 - xy^2 - z$, find the normal vector $\boldsymbol{u}$ that maximizes the directional derivative of $f(x, y, z)$ in the direction of $\boldsymbol{u}$ at point $(1, 1, 0)$.

Solution

From the last problem we know that $\nabla f(1,1,0) = [2,-2,-1]$, Hence, the directional derivative of $f(1,1,0)$ is maximized in the direction of the normal vector

$$\boldsymbol{u} = \frac{[2,-2,-1]}{|[2,-2,-1]|} = \frac{1}{3}[2,-2,-1].$$

Problem 8. Gradient of a product

Let $f(x, y)$ and $g(x, y)$ be scalar functions. Prove:

$$\nabla(fg) = f \cdot \nabla g + g \cdot \nabla f.$$

Solution

We will drive $\nabla(fg)$ and $f \cdot \nabla g + g \cdot \nabla f$ separately to prove that they are equal.

First:

$$\begin{aligned} \nabla(fg) &= \left[\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y},\right] \\ &= \left[g \cdot \frac{\partial f}{\partial x} + f \cdot \frac{\partial g}{\partial x}, g \cdot \frac{\partial f}{\partial y} + f \cdot \frac{\partial g}{\partial y}\right] \end{aligned}$$

Solution

On the other hand,

$$\begin{aligned} f\nabla g + g\nabla f &= f \cdot \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y},\right] + g \cdot \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y},\right] \\ &= \left[f \cdot \frac{\partial g}{\partial x}, f \cdot \frac{\partial g}{\partial y},\right] + \left[g \cdot \frac{\partial f}{\partial x}, g \cdot \frac{\partial f}{\partial y},\right] \\ &= \left[g \cdot \frac{\partial f}{\partial x} + f \cdot \frac{\partial g}{\partial x}, g \cdot \frac{\partial f}{\partial y} + f \cdot \frac{\partial g}{\partial y}\right]. \end{aligned}$$

This completes the proof.