ENGG1410-F Tutorial: Determinant by Permutation Inversions

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In the lecture, we defined matrix determinant in a recursive manner. Today we will see an alternative definition. This definition is more fundamental, but perhaps less intuitive, and definitely more difficult to apply when it comes to determinant calculation.

$$
\begin{aligned}
\begin{vmatrix} x_{11} \end{vmatrix} &= x_{11} \\
\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} &= x_{11}x_{22} - x_{12}x_{21} \\
\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} &= x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + \\
& \quad x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}
\end{aligned}
$$

Any patterns?

$$\begin{vmatrix} x_{11} \end{vmatrix} = x_{11}$$

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

Pattern 1: $n!$ terms (for an $n \times n$ matrix).

$$\begin{vmatrix} x_{11} \end{vmatrix} = x_{11}$$

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

Pattern 1: $n!$ terms (for an $n \times n$ matrix).

Pattern 2: In each term, the three elements are in different rows and columns.

In general, given an $n \times n$ matrix $\boldsymbol{A}$, the determinant $det(\boldsymbol{A})$ includes $n!$ terms, where each term is the product of n elements in $\boldsymbol{A}$ at distinct rows and columns.

Remaining question: $+$ or $-$ for each term?

$$\begin{vmatrix} x_{11} \end{vmatrix} = x_{11}$$

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

Observe that we have arranged the elements of each term in this order: row 1, row 2, ..., row n.

$$\begin{vmatrix} x_{11} \end{vmatrix} = x_{11}$$

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

Observe that we have arranged the elements of each term in this order: row 1, row 2, ..., row n.
Also observe that in each term the sequence of red numbers is a **permutation** of $1, 2, ..., n$.

- E.g., for the 3×3 matrix, the red numbers in the 6 terms represent these permutations (from left to right): $123, 132, 213, 231, 312, 321$.

In general, given an $n \times n$ matrix $\boldsymbol{A}$, the determinant $det(\boldsymbol{A})$ includes $n!$ terms, where each term is the product of n elements in $\boldsymbol{A}$ at distinct rows and columns.

Specifically, each term has the form $x_{1,p_1} x_{2,p_2} ... x_{n,p_n}$, where the sequence $p_1 p_2, ... p_n$ is a permutation of $1, 2, ..., n$. Furthermore, each term represents a different permutation.

Remaining question: $+$ or $-$ for each term?

Consider permutation 132.

- $3, 2$ form an inversion pair, because $3 > 2$ yet 3 stands before 2.
- This is the only inversion pair in the permutation.

Consider permutation 312.

- This permutation has two inversion pairs: $(3, 2)$, $(3, 1)$.

Consider permutation 321.

- This permutation has three inversion pairs: $(3, 2)$, $(3, 1)$, $(2, 1)$.

$$\begin{vmatrix} x_{11} \end{vmatrix} = x_{11}$$

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31}$$

Observe that the sign of each term is:

- $+$ if the permutation in the term has an even number of inversion pairs;
- $-$ if the permutation in the term has an odd number of inversion pairs.

In general, given an $n \times n$ matrix $\boldsymbol{A}$, the determinant $det(\boldsymbol{A})$ includes $n!$ terms, where each term is the product of n elements in $\boldsymbol{A}$ at distinct rows and columns.

Specifically, each term has the form $x_{1,p_1} x_{2,p_2} ... x_{n,p_n}$, where the sequence $p_1 p_2, ... p_n$ is a permutation of $1, 2, ..., n$. Furthermore, each term represents a different permutation.

The sign of each term is:

- $+$ if the permutation in the term has an even number of inversion pairs;
- $-$ if the permutation in the term has an odd number of inversion pairs.

This gives the full definition of determinant we aim to introduce in this talk.

The definition in the preceding slide is typically difficult to apply if you want to calculate the determinant. But there are exceptions. We will see such an example in the next slide.

What is the determinant of the next matrix? Why?

$$\begin{vmatrix} 87732 & 2348 & 239 & 93 & 29083 \\ 8732 & 348 & 39 & 1933 & 9053 \\ 382 & 484 & 0 & 0 & 0 \\ 882 & 1484 & 0 & 0 & 0 \\ 8827 & 14845 & 0 & 0 & 0 \end{vmatrix}$$