THE CHINESE UNIVERSITY OF HONG KONG

ENGG1410(A-F) Linear Algebra and Vector Calculus for Engineers

Mid-term Examination

Instructions

- This paper must be answered in **English**. The duration of the exam is **1.5 hours**.
- Attempt **all** questions. The full mark is **100**.
- Write all your answers in the **answer book**.
- All matrices in this paper contain only **real values**.
- You need to show the **details** of your work.
- **Do not take this paper away.** The paper must be returned at the end of the exam.

Question 1 (12 marks). Consider the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ -1 & -3 & -2 & 1 & 1 \\ 2 & 0 & 6 & 1 & 1 \\ 0 & 2 & -1 & 2 & 1 \\ 0 & 2 & -1 & 2 & 1 \end{bmatrix}.$$

(a) (10 marks) Convert it into the row echelon form using row elementary operations.

(b) (2 marks) Find the rank of $\boldsymbol{A}$.

Question 2 (15 marks).

(a) (5 marks) Use Gauss elimination to solve the following linear system (note that you must give all possible solutions):

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

(b) (7 marks) Find all values of a and b such that the following linear system has a unique solution for x_1, x_2, x_3:

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 + 4x_3 &= 9 \\ 3x_1 - x_2 + a \cdot x_3 &= b \end{aligned}$$

(c) (3 marks) Find all values of a and b such that the linear system in (b) has no solutions.

Question 3 (17 marks).

(a) (10 marks) Consider the following two matrices

$$\boldsymbol{A} = \begin{bmatrix} 6 & 1 & 10 \\ -12 & 1 & -14 \\ -36 & 3 & -40 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 6 & 1 & 8 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Compute the following determinants: $det(\boldsymbol{A})$, $det(\boldsymbol{B})$ and $det(\boldsymbol{A}^{-1}\boldsymbol{B}^{\top})$

(b) (7 marks) Use Cramer's rule to solve the following linear system:

$$\begin{aligned} x_1 - x_2 &= 9 \\ -x_1 + x_2 + x_3 &= 5 \\ x_2 - x_3 &= 8 \end{aligned}$$

Question 4 (13 marks). Compute $(\frac{1}{2}\boldsymbol{A})^{-1}$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

Question 5 (10 marks). Consider an arbitrary 2×2 matrix:

$$\boldsymbol{A} \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

which has real eigenvalues of λ_1 and λ_2 (in the special case where $\boldsymbol{A}$ has only one distinct eigenvalue, $\lambda_1 = \lambda_2$). Prove: $a + d = \lambda_1 + \lambda_2$.

Question 6 (17 marks). It is known that matrix

$$\boldsymbol{A} \quad = \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

can be diagonalized into the form $\boldsymbol{PBP}^{-1}$. Give the details of $\boldsymbol{P}$ and $\boldsymbol{B}$.

Question 7 (16 marks). Recall that an $n \times n$ matrix $\boldsymbol{A} = [a_{ij}]$ is skew-symmetric if $a_{ij} = -a_{ji}$ for all $i, j \in [1, n]$. Answer the following questions.

(a) (10 marks) Let

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Find all possible values a, b, c, d that make $\boldsymbol{A}$ skew-symmetric and orthogonal.

(b) (6 marks) Let

$$\boldsymbol{B} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

be an orthogonal matrix. Prove: $\boldsymbol{B}$ cannot be skew-symmetric.