Lecture Notes: Surfaces, Tangent Planes, and Surface Normals

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1 Surfaces

We will focus on $\mathbb{R}^3$ with dimensions x, y, and z. Consider a plane $x + 2y + 3z - 4 = 0$, or a sphere $x^2 + y^2 + z^2 = 1$. In mathematics, we call them "surfaces".

Formally, a *surface* can be defined by equating scalar function $f(x, y, z)$ to 0, namely, $f(x, y, z) = 0$. In the plane example, $f(x, y, z) = x + 2y + 3z - 4$, whereas in the sphere example, $f(x, y, z) = x^2 + y^2 + z^2 - 1$. It would be helpful to understand why $f(x, y, z) = 0$ is a surface in the following way. Take a point (x, y) in the xy plane, and solve the value of z from $f(x, y, z) = 0$. If z exists, think of z as the "elevation" of a mountain at the longitude x and altitude y. If you move (x, y) around, using z you will be tracing out the top of an undulating mountain. Note that sometimes multiple z may satisfy $f(x, y, z) = 0$, as is true for the sphere $x^2 + y^2 + z^2 = 1$.

2 Tangent Planes and Surface Normals

Consider a surface $f(x, y, z) = 0$. Fix a point $p(x_0, y_0, z_0)$ on the surface such that $\frac{\partial f}{\partial x}(x_0, y_0, z_0)$, $\frac{\partial f}{\partial y}(x_0, y_0, z_0)$, $\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ exist, and are not all equal to 0 — note that $[\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z)]$ is exactly the gradient of the scale function $f(x, y, z)$.

Take an *arbitrary* curve C on the surface passing p. We know that C can be described by functions $x(t), y(t)$, and $z(t)$, which take a real-valued parameter t, and give the x-, y-, and z-coordinates of a point on C. Let t_0 be the value of t corresponding to p (hence, $x_0 = x(t), y_0 = y(t)$, and $z_0 = z(t)$). We assume that $x'(t_0)$, $y'(t_0)$, $z'(t_0)$ exist, and are not all equal to 0.

As C is on the surface, we know that

$$f(x(t), y(t), z(t)) = 0.$$

Taking the derivative of both sides with respect to t gives:

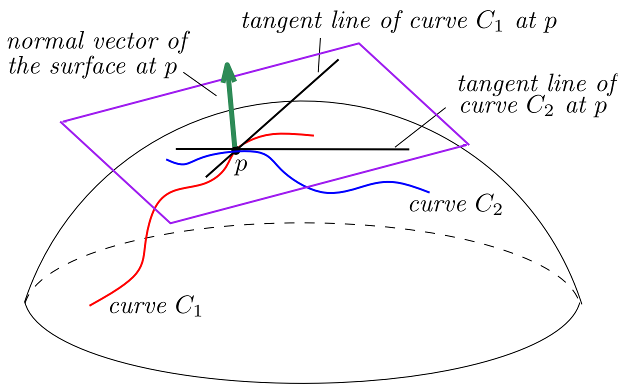
$$\begin{aligned}
\frac{d\Big(f(x(t), y(t), z(t))\Big)}{dt} &= 0 \quad \Rightarrow \\
\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} &= 0 \quad \text{(applied the chain rule here)} \Rightarrow \\
\Big[\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z)\Big] \cdot \Big[x'(t), y'(t), z'(t)\Big] &= 0 \quad \Rightarrow \\
\nabla f(x, y, z) \cdot \Big[x'(t), y'(t), z'(t)\Big] &= 0.
\end{aligned}$$

Applying the above equation to point p results in

$$\nabla f(x_0, y_0, z_0) \cdot \Big[x'(t_0), y'(t_0), z'(t_0)\Big] = 0$$

The above equation tells us something interesting. Notice that $[x'(t_0), y'(t_0), z'(t_0)]$ is a tangent vector of C at p. By our assumptions, neither $\nabla f(x_0, y_0, z_0)$ nor $[x'(t_0), y'(t_0), z'(t_0)]$ is $\mathbf{0}$. Therefore, the direction of $\nabla f(x_0, y_0, z_0)$ is *perpendicular* to that of $[x'(t_0), y'(t_0), z'(t_0)]$.

Here is something even more interesting. Recall that we chose C as an *arbitrary* curve passing p whose tangent vector at p is not $\mathbf{0}$. There can be an infinite number of such curves (the figure below shows two examples). *All* their tangent lines must be perpendicular to the direction of $\nabla f(x_0, y_0, z_0)$! It thus follows that *all those tangent lines must form a plane*, and that the direction of $\nabla f(x_0, y_0, z_0)$ is perpendicular to this plane!



The plane aforementioned is therefore called the *tangent plane* of the surface at p. $\nabla f(x_0, y_0, z_0)$ is called a *normal vector* of the surface at p.

Example. Consider the sphere $x^2 + y^2 + z^2 = 1$ and a point $p(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}})$ on the sphere. What is the tangent plane π of the sphere at p?

Define $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Hence, the sphere is given by $f(x, y, z) = 0$. The gradient of f is:

$$\begin{aligned}\nabla f(x, y, z) &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \\ &= [2x, 2y, 2z].\end{aligned}$$

From the earlier discussion, we know that π must be perpendicular to the gradient vector at p, namely: $\nabla f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}) = [\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{6}}]$. To write out the equation for π, let $q = (x, y, z)$ be any point on π. We know that the vector $\overrightarrow{pq}$ must be perpendicular to $\nabla f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}})$ (which is a normal vector of π at p). This means:

$$\begin{aligned}\overrightarrow{pq} \cdot \nabla f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}) &= 0 \quad \Rightarrow \\ \left[x - \frac{2}{\sqrt{2}}, y - \frac{2}{\sqrt{3}}, z - \frac{2}{\sqrt{6}}\right] \cdot \left[\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{6}}\right] &= 0 \quad \Rightarrow \\ \frac{2}{\sqrt{2}}\left(x - \frac{2}{\sqrt{2}}\right) + \frac{2}{\sqrt{3}}\left(y - \frac{2}{\sqrt{3}}\right) + \frac{2}{\sqrt{6}}\left(z - \frac{2}{\sqrt{6}}\right) &= 0 \quad \Rightarrow \\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{6}} &= 2.\end{aligned}$$

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