Lecture Notes: Path Independence

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

Let C be a piecewise-smooth curve from point $p = (0,0)$ to $q = (1,1)$ in $\mathbb{R}^2$. Would you be able to calculate the following line integral?

$$\int_C y\,dx + x\,dy. \tag{1}$$

You may sense that something is missing: the details of C have not been given yet! It turns out that *we do not need those details* to evaluate the integral. In other words, the result of the integral will *always* be the same regardless of C. Furthermore, introducing $g(x,y) = xy$, we will learn that the result of (1) is definitely $g(1,1) - g(0,0) = 1$!

Line integrals such as (1) constitute a *path independent* set. In this lecture, we will study this interesting type of line integrals.

1 Path Independence in $\mathbb{R}^2$

Definition 1. *Fix scalar functions $f_1(x,y)$ and $f_2(x,y)$. Define S to be the set of all possible line integrals of the form*

$$\int_C f_1\,dx + f_2\,dy$$

where C can be any piecewise smooth arc with a starting point and an ending point. S is **path independent** *if*

$$\int_{C_1} f_1\,dx + f_2\,dy \;=\; \int_{C_2} f_1\,dx + f_2\,dy.$$

holds for any C_1 and C_2 in S that share the same starting and ending points.

The following theorem gives a convenient way to judge whether S is path independent.

Theorem 1. *Suppose that $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are both continuous in $\mathbb{R}^2$. S is path independent if and only if*

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}. \tag{2}$$

Proof. See Appendix A. □

Example 1. Consider the set S of line integrals of the form:

$$\int_C y\,dx + x\,dy$$

where C is a piecewise smooth arc in $\mathbb{R}^2$. By Theorem 1, we know that S is path independent because $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 1$, where $f_1 = y$ and $f_2 = x$.

Now, consider S instead to be the set of line integrals of the form

$$\int_C y^2(\sin(x) + x \cdot \cos(x))\, dx + 2xy \sin(x)\, dy$$

where C is a piecewise smooth arc in $\mathbb{R}^2$. Here, $f_1 = y^2(\sin(x) + x \cdot \cos(x))$ and $f_2 = 2xy\sin(x)$. Since $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 2y(\sin(x) + x \cdot \cos(x))$, S is path independent. □

2 Line Integral Evaluation under Path Independence: Method 1

After realizing path independence, we may choose to evaluate a line integral along a simpler arc to make calculation easier, as is demonstrated in the example below.

Example 2. Suppose that we want to calculate

$$\int_C y\, dx + x\, dy$$

where C is the arc from $(0, 0)$ to $(1, 1)$ on the curve $\boldsymbol{r}(t) = [\frac{4}{\pi^2} t^2 \sin t, \frac{2}{\pi e^{\pi/2}} t \cdot e^t]$. Note that the curve is deliberately chosen to be complicated to make it difficult and tedious to compute the integral using the methods taught in previous lectures.

On the other hand, Example 1 tells us that the value of the integral will not be affected if we replace C with any piecewise smooth arc C' from $(0, 0)$ to $(1, 1)$. Let us choose C' to be the straight line segment directed from $(0, 0)$ to $(1, 1)$. In other words, C' is an arc on the curve $\boldsymbol{r'}(t) = [t, t]$ defined by increasing t from 0 to 1. Hence:

$$\begin{aligned} \int_C y\, dx + x\, dy &= \int_0^1 t\frac{dx}{dt} dt + t\frac{dy}{dt} dt \\ &= \int_0^1 2t\ dt = 1 \end{aligned}$$

□

3 Line Integral Evaluation under Path Independence: Method 2

In this section, we will introduce another method to evaluate a line integral based on path independence. This method requires us to figure out an "original function". Once we have managed to do so, the line integral becomes trivial to evaluate.

Let us start by stating an important theorem:

Theorem 2. *Fix scalar functions $f_1(x, y)$ and $f_2(x, y)$. Suppose that $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are both continuous in $\mathbb{R}^2$. If*

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \tag{3}$$

then there exists a function $g(x, y)$ such that

$$f_1(x, y) = \frac{\partial g}{\partial x}, \text{ and } f_2(x, y) = \frac{\partial g}{\partial y}. \tag{4}$$

Proof. See Appendix B. □

Example 3. Consider $f_1 = y$ and $f_2 = x$. Since $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 1$, Theorem 2 asserts that there must be an "original function" $g(x, y)$ satisfying $f_1(x, y) = \frac{\partial g}{\partial x}$ and $f_2(x, y) = \frac{\partial g}{\partial y}$. For example, one such function is $g(x, y) = xy$.

Consider instead $f_1 = y^2(\sin(x) + x \cdot \cos(x))$ and $f_2 = 2xy \sin(x)$. Since $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = 2y(\sin(x) + x \cdot \cos(x))$, we must be able to find an "original function" $g(x, y)$, an example of which is $g(x, y) = x \sin(x) \cdot y^2$. □

Now consider a path-independent set S of line integrals of the form

$$\int_C f_1\, dx + f_2\, dy.$$

By Theorem 1, f_1 and f_2 obey (2). Let $g(x, y)$ be any "original function" promised by Theorem 2. For simplicity, we may also write $g(x, y)$ as $g(p)$ where p is the point (x, y). The next theorem provides an extremely simple way to evaluate a line integral from S:

Theorem 3. *Suppose that C is an arc from point p to point q. Then:*

$$\int_C f_1\, dx + f_2\, dy \quad = \quad g(q) - g(p).$$

Proof. See Appendix C. □

Example 4. Let us reconsider the line integral in Example 1. Since there is an original function $g(x, y) = xy$ satisfying $f_1(x, y) = \frac{\partial g}{\partial x}$ and $f_2(x, y) = \frac{\partial g}{\partial y}$, Theorem 3 tells us that the value of the line integral can be obtained immediately as $g(1, 1) - g(0, 0) = 1$. □

Example 5. Suppose that we want to calculate:

$$\int_C y^2(\sin(x) + x \cdot \cos(x))\, dx + \int_C 2xy \sin(x)\, dy. \tag{5}$$

where C is the arc from $(0, 0)$ to $(1, 1)$ on the curve $\boldsymbol{r}(t) = [\frac{4}{\pi^2} t^2 \sin t, \frac{2}{\pi e^{\pi/2}} t \cdot e^t]$.

Define $f_1 = y^2(\sin(x) + x \cos(x))$ and $f_2 = 2xy \sin(x)$. Since there exists $g(x, y) = x \sin(x) \cdot y^2$ satisfying $\frac{\partial g}{\partial x} = f_1$ and $\frac{\partial g}{\partial y} = f_2$, we know by Theorem 3 that (5) $= g(1, 1) - g(0, 0) = \sin(1)$. □

4 Path Independence in $\mathbb{R}^d$

The discussion in the above sections can be generalized to $\mathbb{R}^d$ for an arbitrary d. Fix d scalar functions $f_1(x_1, x_2, ..., x_d)$, $f_2(x_1, x_2, ..., x_d)$, ..., and $f_d(x_1, x_2, ..., x_d)$. Define S to be the set of all possible line integrals of the form

$$\int_C f_1\, dx_1 + f_2\, dx_2 + ... + f_d\, dx_d$$

where C is a piecewise smooth arc in $\mathbb{R}^d$.

Definition 2. *We say that S is* **path independent** *if*

$$\int_{C_1} f_1\,dx_1 + f_2\,dx_2 + ... + f_d\,dx_d \;=\; \int_{C_2} f_1\,dx_1 + f_2\,dx_2 + ... + f_d\,dx_d$$

holds for any two piecewise-smooth arcs C_1 and C_2 in $\mathbb{R}^d$ that share the same starting and ending points.

We state the next theorem without proof:

Theorem 4. *S is path independent if and only if we can find a function $g(x_1, x_2, ..., x_d)$ such that*

$$\begin{aligned} f_1(x_1, .., x_d) &= \frac{\partial g}{\partial x_1}(x_1, .., x_d) \\ f_2(x_1, .., x_d) &= \frac{\partial g}{\partial x_2}(x_1, .., x_d) \\ &\cdots \\ f_d(x_1, .., x_d) &= \frac{\partial g}{\partial x_d}(x_1, .., x_d). \end{aligned}$$

When S is path independent, for any points $p = (x_{p_1}, x_{p_2}, ..., x_{p_d})$, $q = (x_{q_1}, x_{q_2}, ..., x_{q_d})$, and any piecewise-smooth curve C from p to q, it holds that

$$\int_C f_1\,dx_1 + f_2\,dx_2 + ... + f_d\,dx_d \;=\; g(x_{q_1}, x_{q_2}, ..., x_{q_d}) - g(x_{p_1}, x_{p_2}, ..., x_{p_d}).$$

Example 6. Let C be a piecewise smooth curve from point $p = (2, 3, 4)$ to $q = (1, 1, 1)$ in $\mathbb{R}^3$, but the other details of C are hidden from you. Calculate:

$$\int_C 2xy^2z\,dx + 2x^2yz\,dy + x^2y^2\,dz. \tag{6}$$

Solution. Let $g(x, y, z) = x^2y^2z$. Clearly, $\frac{\partial g}{\partial x} = 2xy^2z$, $\frac{\partial g}{\partial y} = 2x^2yz$, and $\frac{\partial g}{\partial z} = 2x^2y^2$. Hence, by Theorem 4, (6) $= g(2, 3, 4) - g(1, 1, 1) = 143$. □

A Proof of Theorem 1

The If-Direction. We will first prove that if (2) holds, then S is path independent. Consider any C_1 and C_2 in S that have the same starting point p and ending point q, as shown in Figure 1a. Let us reverse the direction of C_2, and thereby obtain a curve C_2' from q to p; see Figure 1b. We will prove that

$$\left(\int_{C_1} f_1\,dx + f_2\,dy\right) + \left(\int_{C_2'} f_1\,dx + f_2\,dy\right) = 0 \tag{7}$$

which indicates that $\int_{C_1} f_1\,dx + f_2\,dy = -\int_{C_2'} f_1\,dx + f_2\,dy = \int_{C_2} f_1\,dx + f_2\,dy$, as desired.

We resort to Green's theorem in proving (7). Consider the closed curve C that concatenates C_1 and C_2'. Let D be the area enclosed by C. Thus,

$$\begin{aligned} (7) &= \int_C f_1\,dx + f_2\,dy \\ \text{(by Green's Theorem)} &= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dxdy \\ \text{(by (2))} &= 0 \end{aligned}$$

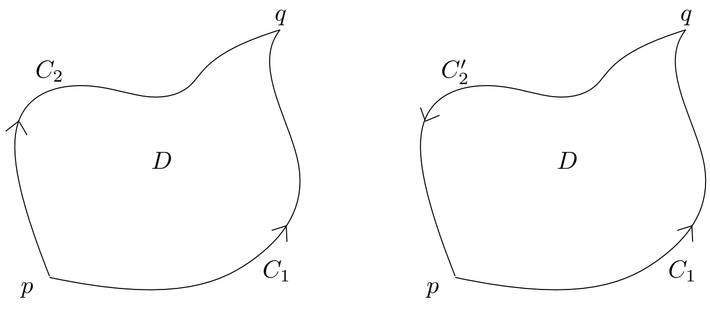

Figure 1: Proof of the if-direction of Theorem 1

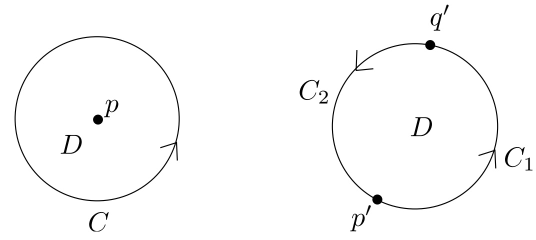

Figure 2: Proof of the only-if direction of Theorem 1

as claimed.

The Only-If Direction. Next, we will prove that if S is path independent, then (2) definitely holds everywhere in $\mathbb{R}^2$. Assume, on the contrary that, this was not true at some point $p = (x_0, y_0)$. Without loss of generality, suppose that

$$\frac{\partial f_2}{\partial x}(x_0, y_0) - \frac{\partial f_1}{\partial y}(x_0, y_0) > 0.$$

As both $\frac{\partial f_2}{\partial x}$ and $\frac{\partial f_1}{\partial y}$ are continuous, so is $\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$. Hence, we can find a small disc D centered at p such that

$$\frac{\partial f_2}{\partial x}(x', y') - \frac{\partial f_1}{\partial y}(x', y') > 0 \tag{8}$$

holds for any point (x', y') in D; see Figure 2a: Let C be the boundary of D, namely, a circle, in the counterclockwise direction. Now consider the following line integral:

$$\int_C f_1\, dx + f_2\, dy \tag{9}$$

which is supposed to be 0 because S is path independent. To see this, choose two distinct points p' and q' on C arbitrarily, and consider the arc C_1 from p' to q' counterclockwise, and the arc C_2 from q' to p' counterclockwise; see Figure 2b. S being path independent implies that

$$\left(\int_{C_1} f_1\, dx + f_2\, dy\right) + \left(\int_{C_2} f_1\, dx + f_2\, dy\right) = 0$$

which directly indicates that (9) = 0.

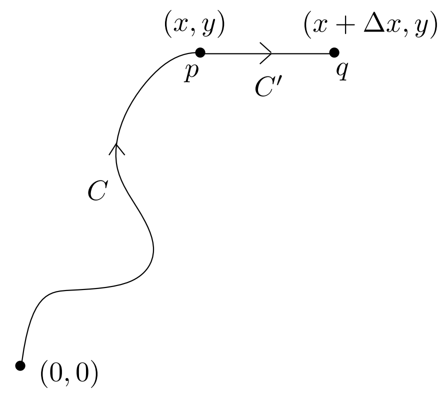

Figure 3: Proof of Theorem 2

However, by Green's Theorem, we have:

$$\begin{aligned} (9) &= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\ \text{(by (8))} &> 0 \end{aligned}$$

which gives a contradiction.

B Proof of Theorem 2

We will construct such a function explicitly:

$$g(x,y) = \int_C f_1\, dx + f_2\, dy$$

where C is any piecewise smooth curve from the origin to the point (x,y). Under the condition (3), Theorem 1 tells us that the value of the integral does not depend on the choice of C. Next, we will show that $g(x,y)$ satisfies (4). Due to symmetry, we will show only $f_1(x,y) = \frac{\partial g}{\partial x}$.

By definition of partial derivative:

$$\frac{\partial g}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x, y) - g(x,y)}{\Delta x}$$

Let p and q denote the points (x,y) and $(x+\Delta x, y)$, respectively. Denote by C' the horizontal directed segment from p to q. See Figure 3. Define C'' as the arc that concatenates C and C'. By definition we have:

$$g(x+\Delta x, y) = \int_{C''} f_1\, dx + f_2\, dy$$

and hence:

$$\begin{aligned} g(x+\Delta x, y) - g(x,y) &= \left(\int_{C''} f_1\, dx + f_2\, dy\right) - \left(\int_C f_1\, dx + f_2\, dy\right) \\ &= \int_{C'} f_1\, dx + f_2\, dy \\ \text{(as y-coordinate does not change on } C') &= \int_{C'} f_1\, dx \\ &= \int_x^{x+\Delta x} f_1\, dx. \end{aligned}$$

The continuity of $\frac{\partial f_1}{\partial x}$ implies that f_1 is continuous on x. Therefore, the mean value theorem tells us that there exists a value $x' \in [x, x+\Delta x]$ such that the above integral equals $f_1(x', y) \cdot \Delta x$. Therefore:

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{g(x+\Delta x, y) - g(x, y)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{f_1(x', y) \cdot \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} f_1(x', y) = f_1(x, y) \end{aligned}$$

which completes the proof.

C Proof of Theorem 3

Suppose that C is an arc on the curve $[x(t), y(t)]$ defined by increasing t from t_p to t_q, namely, $p = (x(t_p), y(t_p))$ and $p = (x(t_q), y(t_q))$. We have

$$\begin{aligned} \int_C f_1\, dx + f_2\, dy &= \int_C \frac{\partial g}{\partial x}\, dx + \int_C \frac{\partial g}{\partial y}\, dy \\ &= \int_{t_p}^{t_q} \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}\, dt \\ &= \int_{t_p}^{t_q} \frac{dg}{dt}\, dt \\ &= g(x(t), y(t))\Big|_{t_p}^{t_q} \\ &= g(q) - g(p). \end{aligned}$$