Exercises: Dot Product and Cross Product

Problem 1. For the following directed segments, give the vectors they define:

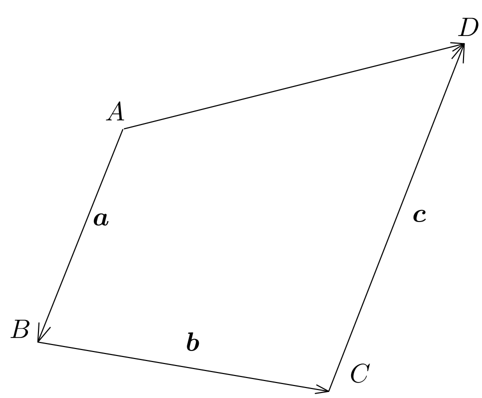
1. $\overrightarrow{(1,2),(2,3)}$
2. $\overrightarrow{(10,20),(11,21)}$
3. $\overrightarrow{(1,-2),(2,3)}$
4. $\overrightarrow{(1,-2,0),(2,3,10)}$

Problem 2. In each of the following cases, indicate whether $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction (i.e., whether their angle is 0):

1. $\boldsymbol{a} = [1,1], \boldsymbol{b} = [2,2]$
2. $\boldsymbol{a} = [1,2,3], \boldsymbol{b} = [20,40,60]$
3. $\boldsymbol{a} = [1,2,3], \boldsymbol{b} = [2,-4,6]$

Problem 3. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be 2d vectors such that $\boldsymbol{a} + \boldsymbol{b} = [3,5]$, and $\boldsymbol{a} - \boldsymbol{b} = [4,6]$. What are $\boldsymbol{a}$ and $\boldsymbol{b}$?

Problem 4. Let A, B, C, D be 4 points in $\mathbb{R}^d$. Suppose that directed segments $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CD}$ define vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively; see the figure below. Prove that $\overrightarrow{AD}$ is an instantiation of $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$.



Problem 5. Give the result of $\boldsymbol{a} \times \boldsymbol{b}$ for each of the following:

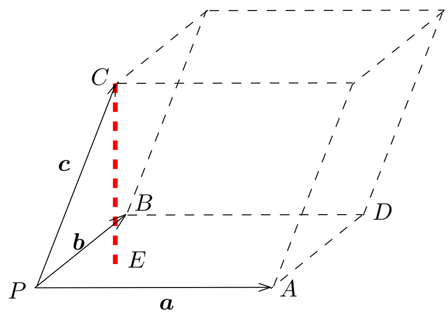
1. $\boldsymbol{a} = [1,2,3], \boldsymbol{b} = [3,2,1]$.
2. $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{b} = [3,2,1]$.

Problem 6. In each of the following, you are given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Give the value of $\cos\gamma$, where γ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

1. $\boldsymbol{a} = [1, 2], \boldsymbol{b} = [2, 5]$
2. $\boldsymbol{a} = [1, 2, 3], \boldsymbol{b} = [3, 2, 1]$

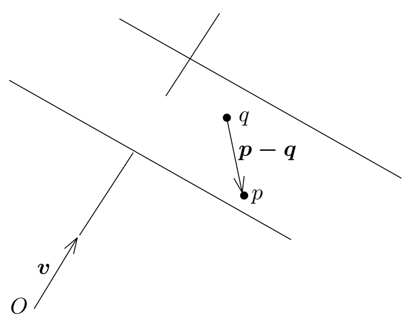
Problem 7. This exercise explores the usage of dot product for calculation of projection lengths. Consider points $P(1, 2, 3), A(2, -1, 4), B(3, 2, 5)$. Let ℓ be the line passing P and A. Now, let us project point B onto ℓ; denote by C the projection. Calculate the distance between P and C.

Problem 8. Let $\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ be directed segments that are not in the same plane. They determine a parallelepiped as shown below:

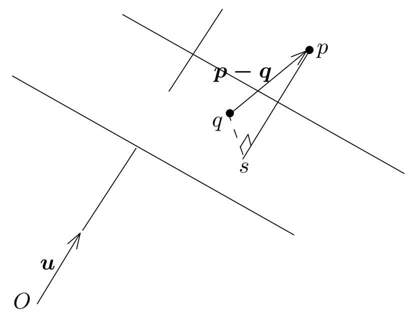


Suppose that $\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ define vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, respectively. Prove that the volume of the parallelepiped equals $|(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}|$.

Problem 9. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{v}$ be a non-zero 3d vector. Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{v}$. Prove that for any p on ρ, it holds that $(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{v} = 0$.



Problem 10. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\boldsymbol{p}$ to denote the corresponding vector $[x, y, z]$. Let q be a point in $\mathbb{R}^3$, and $\boldsymbol{u}$ be a unit 3d vector (i.e., $|\boldsymbol{u}| = 1$). Denote by ρ the plane passing q that is perpendicular to the direction of $\boldsymbol{u}$. Prove that for any p in $\mathbb{R}^3$, its distance to ρ equals $|(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{u}|$.



Problem 11. Consider the plane $x + 2y + 3z = 4$ in $\mathbb{R}^3$. Calculate the distance from point $(0, 0, 0)$ to the plane.