Exercises: Similarity Transformation

Problem 1. Diagonalize the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Problem 2. Consider again the matrix $\boldsymbol{A}$ in Problem 5. Calculate $\boldsymbol{A}^t$ for any integer $t \geq 1$.

Problem 3. Diagonalize the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Problem 4. Suppose that matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar to each other, namely, there exists $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$. Prove: if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ under eigenvalue λ, then $\boldsymbol{P}\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$ under eigenvalue λ.

Problem 5. Suppose that an $n \times n$ matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$. Prove: for any $n \times 1$ vector $\boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{x}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$.

Problem 6. Prove or disprove: if an $n \times n$ matrix $\boldsymbol{A}$ has rank n, then it must have n independent eigenvectors.

Problem 7. Prove that $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Problem 8. Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be three $n \times n$ matrices for some integer n. Prove that if $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ and $\boldsymbol{B}$ is similar to $\boldsymbol{C}$, then $\boldsymbol{A}$ is similar to $\boldsymbol{C}$.

Problem 9. Decide whether

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

is similar to

$$\boldsymbol{B} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$