Exercises: Line Integral by Coordinate

Problem 1. Let C be the curve from point $p = (0, 0)$ to $q = (2, 4)$ on the parabola $y = x^2$. Calculate $\int_C (x^2 - y^2) dx$.

Solution: First, write C into its parametric form: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = t$, and $y(t) = t^2$. Points p and q are given by $t = 0$ and 2, respectively. Thus:

$$\begin{aligned}
\int_C (x^2 - y^2) dx &= \int_0^2 (t^2 - t^4) \frac{dx}{dt} dt \\
&= \int_0^2 (t^2 - t^4) dt \\
&= 8/3 - 32/5.
\end{aligned}$$

Problem 2. Let $\boldsymbol{r}(t) = [t, t^2, t^3]$ and $\boldsymbol{f}(x, y, z) = [x - y, y - z, z - x]$. Let C be the curve from the point of $t = 0$ to the point of $t = 1$. Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$.

Solution:

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{f}(\boldsymbol{r}) \cdot \boldsymbol{r}'(t)\, dt \\
&= \int_0^1 [t - t^2, t^2 - t^3, t^3 - t] \cdot [1, 2t, 3t^2]\, dt \\
&= \int_0^1 t - t^2 + 2t^3 - 2t^4 + 3t^5 - 3t^3\, dt \\
&= \int_0^1 t - t^2 - t^3 - 2t^4 + 3t^5\, dt \\
&= 1/60.
\end{aligned}$$

Problem 3. Same as in Problem 2, except that C is defined by decreasing t from 1 to 0 (i.e., reversing the direction as in Problem 2).

Solution: When the direction of the arc is reversed, the value of the integer integral (by coordinate) is reversed. Hence, the answer is $-1/60$.

Solution:

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{f}(\boldsymbol{r}) \cdot \boldsymbol{r}'(t)\, dt \\
&= \int_0^1 [t - t^2, t^2 - t^3, t^3 - t] \cdot [1, 2t, 3t^2]\, dt \\
&= \int_0^1 t - t^2 + 2t^3 - 2t^4 + 3t^5 - 3t^3\, dt \\
&= \int_0^1 t - t^2 - t^3 - 2t^4 + 3t^5\, dt \\
&= 1/60.
\end{aligned}$$

Problem 4. Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ where $\boldsymbol{f}(x, y) = [y^2, -x^2]$, and C is the arc from $(0, 0)$ to $(1, 4)$ on the curve $y = 4x^2$.

Solution. Let us first represent the curve $y = x^2$ in its parametric form: $\boldsymbol{r}(t) = [t, 4t^2]$. C is defined by increasing t from 0 to 1. Hence:

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{f}(\boldsymbol{r}) \cdot \boldsymbol{r}'(t)\, dt \\
&= \int_0^1 [y(t)^2, -x(t)^2] \cdot [1, 8t]\, dt \\
&= \int_0^1 y(t)^2 - 8t \cdot x(t)^2\, dt \\
&= \int_0^1 (4t^2)^2 - 8t \cdot t^2\, dt \\
&= \int_0^1 16t^4 - 8t^3\, dt \\
&= \; = 6/5.
\end{aligned}$$

Problem 5. Calculate

$$\int_C xy\, dx + x^2 y^2\, dy$$

where C is the quarter-arc from $(1, 0)$ to $(0, 1)$ on the circle $x^2 + y^2 = 1$.

Solution. Let us first represent the circle in its parametric form: $\boldsymbol{r}(t) = [\cos t, \sin t]$. C is defined by increasing t from 0 to $\pi/2$. Hence:

$$\begin{aligned}
\int_C xy\, dx + x^2 y^2\, dy &= \int_0^{\pi/2} \left(xy \frac{dx}{dt} + x^2 y^2 \frac{dy}{dt} \right) dt \\
&= \int_0^{\pi/2} \big(\cos t \sin t \cdot (-\sin t) + (\cos^2 t)(\sin^2 t) \cos t \big)\, dt \\
&= -\int_0^{\pi/2} \sin^4 t \cos t\, dt \\
&= -\int_0^{\pi/2} \sin^4 t \;\; d(\sin t) \\
&= -1/5.
\end{aligned}$$

Problem 6. Let $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Let p be the point given by $t = \pi/4$. Calculate $\frac{dx}{ds}$ at p.

Solution:

$$\begin{aligned}\frac{dx}{ds} &= \frac{dx/dt}{ds/dt}\\ &= \frac{dx/dt}{\sqrt{(dx/dt)^2+(dy/dt)^2}}\\ &= \frac{x'(t)}{\sqrt{(x'(t))^2+(y'(t))^2}}\\ &= \frac{-\sin(t)}{\sqrt{(-\sin(t))^2+(\cos(t))^2}}\\ &= -\sin(t).\end{aligned}$$

Hence, the value of $\frac{dx}{ds}$ at p is $-\sin(\pi/4) = -1/\sqrt{2}$.

Problem 7. Let $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$. Let p be the point given by $t = t_0$. Prove that $[\frac{dx}{ds}(t_0), \frac{dy}{ds}(t_0), \frac{dz}{ds}(t_0)]$ is a unit tangent vector at p.

Proof:

$$\frac{dx}{ds} = \frac{dx/dt}{ds/dt} = \frac{dx/dt}{\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}}$$

Similarly:

$$\begin{aligned}\frac{dy}{ds} &= \frac{dy/dt}{ds/dt} = \frac{dy/dt}{\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}}\\ \frac{dz}{ds} &= \frac{dz/dt}{ds/dt} = \frac{dz/dt}{\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}}.\end{aligned}$$

Therefore:

$$\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right] = \frac{[x'(t), y'(t), z'(t)]}{\sqrt{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}}$$

which proves that $[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}]$ is a tangent vector. Furthermore:

$$\left|\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]\right|^2 = \frac{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2}{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2} = 1$$

which means that $[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}]$ is a unit vector. □

Problem 8. This problem allows you to see the equivalence of line integral by arc length and line integral by coordinate. Let $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Convert $\int_C x\,dx + \int_C y^2\,dy$ to line integral by arc length.

Solution:

$$\begin{aligned}\int_C x\,dx + \int_C y^2\,dy &= \int_C x\,\frac{dx}{ds}ds + \int_C y^2\,\frac{dy}{ds}ds\\ &= \int_C x\,\frac{dx}{ds} + y^2\,\frac{dy}{ds}\,ds \qquad (1)\end{aligned}$$

In Problem 4, we have shown that $\frac{dx}{ds} = -\sin(t) = -y(t)$. Similarly:

$$\begin{aligned}
\frac{dy}{ds} &= \frac{dy/dt}{ds/dt} \\
&= \frac{dy/dt}{\sqrt{(dx/dt)^2 + (dy/dt)^2}} \\
&= \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \\
&= \frac{\cos(t)}{\sqrt{(-\sin(t))^2 + (\cos(t))^2}} \\
&= x(t).
\end{aligned}$$

Hence:

$$(1) \quad = \quad \int_C -xy + y^2 x \, ds.$$