Vertex Covers: Indirect Certificates and New FPT Algorithms

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Outline

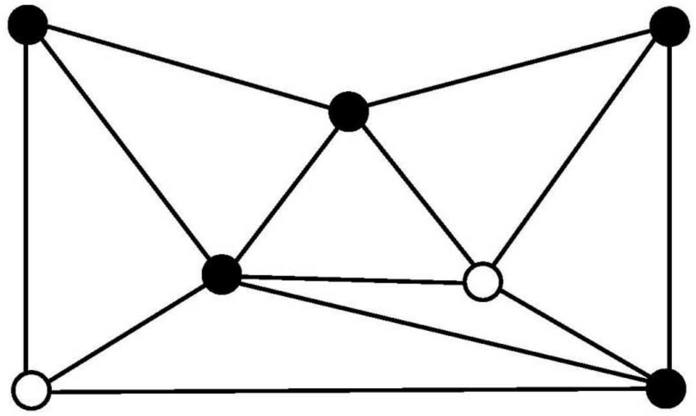
- Introduction
- Indirect certificates
- FPT algorithms
- Conclusion

Introduction

Vertex Cover (NP-complete)

Input: Graph G = (V,E), parameter *k*.

Question: Does G contain *k* vertices that cover all edges?



Parameterized Complexity

Task: Compute 2^n

Direct: O(n^2) time.

Repeated squaring: O($n^{1.59}$) or O($n \log^2 n \log\log n$) time.

Input size: O($\log n$).

Question: Can we do it in polynomial time?

Answer: No, because output size $\Theta(n)$.

Parameterizd Complexity

Input I → Algorithm A → Output O

Time complexity: classical T(|I|) → 2D-way T(|I|, |O|)

Parameterized complexity: T(|I|, *k*)

k: parameter of interest, typically |O|, solution size, or structural parameter (e.g., number of edge deletions to obtain a planar graph).

FPT Algorithms

FPT algorithm: f(k)$n^{O(1)}$ time.

FPT = fixed-parameter tractable

$k^k n$

$4^k k^2 n^2$

$2^k n$

$2^{\sqrt{k}} n$

$n^2 + 2^k$

$1.2738^k + kn$

To solve NP-hard problems effectively

for relatively small *k*.

FPT Algorithms

	k	*10*	*20*	*50*	*100*	$n = 1000$
DS	n^k	10^{30}	10^{60}	10^{150}	10^{300}	
IS	$n^{0.8k}$	10^{24}	10^{48}	10^{120}	10^{240}	
VC	$2^k n$	10^6	10^9	10^{18}	10^{33}	
	$1.2738^k + kn$	10^4	10^4	10^5	10^{10}	

Vertex Cover, Clique, Independent Set:

No problem to obtain optimal solutions for graphs with 200 vertices!

Vertex Cover

Input: Graph G = (V,E), parameter *k*.

Question: Does G contain *k* vertices that cover all edges?

Task: FPT algorithms for Vertex Cover.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FPT Algorithms for Vertex Cover

Graph minor

Fellow and Langston (1986) $\rightarrow$ O(f(k)n^3)

f(k) astronomical

Johnson (1987) $\rightarrow$ O(f(k)n^2)

f(k) $\approx 2^{2^{500k}}$

Matching

Papadimitriou and Yannakakis (1993) $\rightarrow$ O($3^k kn$)

FPT algorithms for Vertex Cover

Bounded search tree

- For any edge *uv*,
  either *u* or *v* must be in a solution → $O(2^k kn)$
- Path P_3 → $O(1.618^k kn)$
- Vertex of degree at least 3 → $O(1.5^k \mathrm{kn})$
- Chan, Kanj, and Xia (2010) → $O(1.2738^k + kn)$


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Ways to Finish 100M

倒走 拿大顶 飞滚 旋翻 单腿跳 滑雪型转步

西施步 玉环醉酒 扭臀步 太空漫游

凌波虚步 精神病人思路广 猫行 梦游 旋风腿

僵尸跳 济公步 比翼双飞 秧歌摆 小鲜肉步

开车 租人 趟泥步 倒撵猴 乘火箭 喷 打的

快闪 最少能 弹弓 风火轮 交叉迴旋

Introduction: Motivations

大道至简 Greatest truth is simple

- Better understanding
- Training students
- Intellectually challenging

化腐朽为神奇 Do bad things in clever ways

New FPT Algorithm for Vertex Cover

Randomly mark each vertex, output N(M).

M: marked vertices.
N(M): neighbors of marked vertices.

Certificate for Vertex Cover

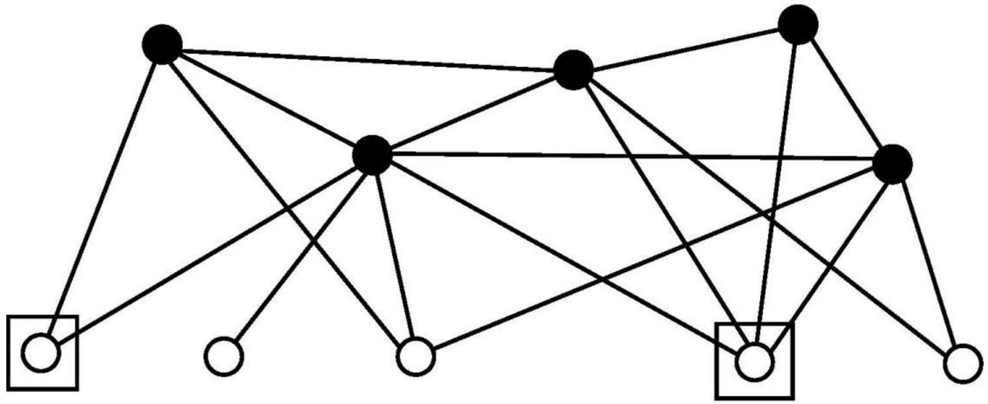
Vertex Cover belongs to class NP.

Natural certificate: solution, i.e., a k-vertex cover X.

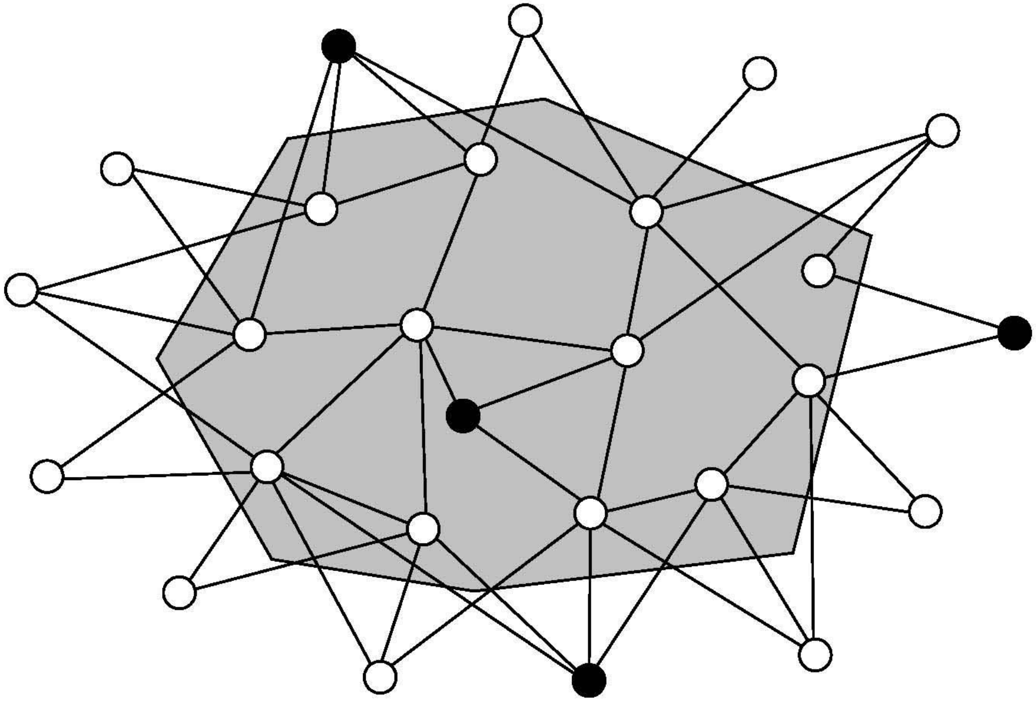
Alternative certificate: subset of X with at most k – log n vertices.

Indirect Certificate

Theorem 1. For any minimal vertex cover X of a graph $G = (V,E)$, $V - X$ contains at most $|X|$ vertices C such that $N(C) = X$.



Indirect Certificate



Indirect Certificate

Given indirect certificate C, we can obtain vertex cover X in linear time.

Therefore C can be used as a certificate to verify that G indeed has a *k* vertex cover.

FPT Algorithm using Indirect Certificate

Partition vertices of G into blue vertices B and red vertices R such that

- B contains vertex cover X, and
- R contains indirect certificate C.

Once we have such a (B,R)-partition, Theorem 1 guarantees that N(R) is a required vertex cover.

How to produce such a (B,R)-partition?

Randomized FPT Algorithm

Algorithm VC-IC

Step 1. Randomly and independently color each vertex either red or blue
with probability ½ to form red vertices R.

Step 2. Return N(R) as a solution.

Theorem 2. Algorithm VC-IC finds, with probability at least 4^{-k},
a *k*-vertex cover of G, if it exists, in O(*m* + *n*) time.

Note: The algorithm can be derandomized by (n,2*k*)-universal sets.

Semi-random Partition

Repeat the following until all vertices of G are coloured:

- Randomly choose an uncoloured vertex *v*, colour it red or blue with probability *p* for red and probability *1-p* for blue, and
- colour all neighbours of *v* blue if *v* is coloured red.

Random Selection

Optimal value for *p* is 1.

Randomly choose a vertex *v* and declare it to be not in solution, and hence put all vertices of N(*v*) into solution.

Random Selection

Algorithm VC-SRP

Step 1. Repeat the following until all vertices are coloured:

Randomly and uniformly choose an uncoloured vertex *v*, colour *v* red and all neighbours of *v* blue to form a (B,R)-partition of V.

Step 2. Output N(R) as X.

Theorem 3. Algorithm VC-SRP finds, with probability at least 2^{-k}, a *k*-vertex cover of G, if it exists, in O(m + n) time.

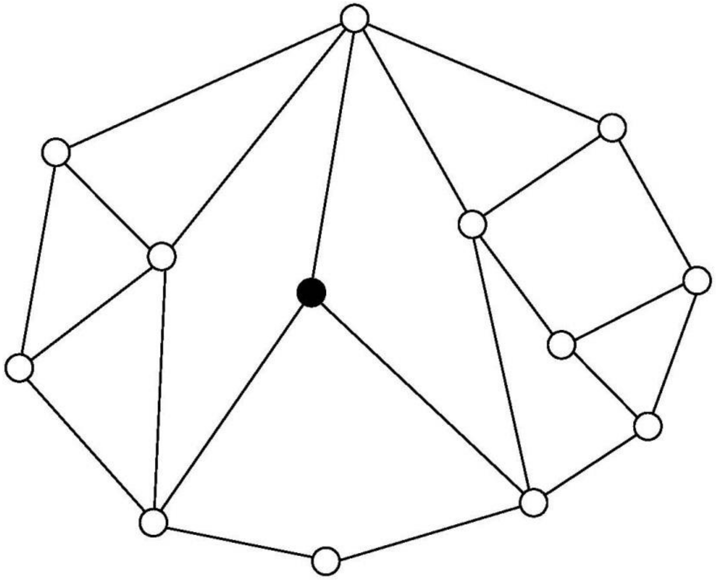
Smaller Indirect Certificate

Theorem 4. Every yes-instance (G,*k*) of Vertex Cover admits an indirect certificate C with at most *k*/3 vertices.

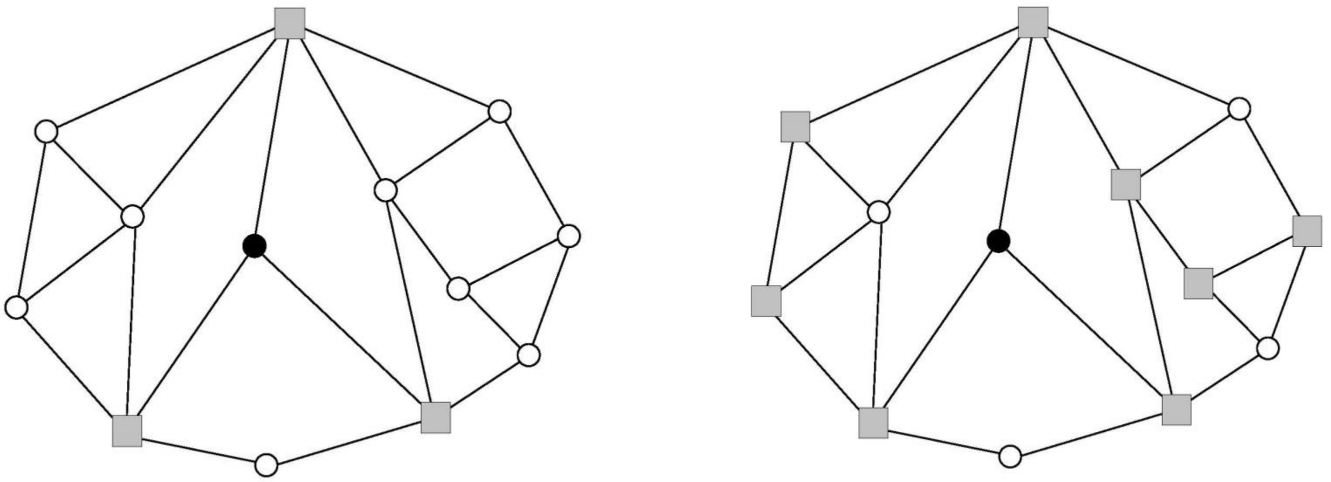
Success probability better

VC-IC: 2.1166^{-k}

VC-SRP: 1.6633^{-k}



Smaller Indirect Certificate



Conclusion

- Indirect certificates are interesting in their own right.
- Potential to use indirect certificates to obtain FPT algorithms.