Grid Decomposition

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This lecture will introduce grid decomposition, which is a fundamental technique for solving many computational geometry problems. We will demonstrate the technique by using it to solve the closest pair and close pairs problems.

Closest Pair and Close Pairs

Let P be a set of points $\mathbb{R}^d$. The objective of the **closest pair problem** is to output a pair of distinct points $p, q \in P$ that have the smallest distance to each other, or formally:

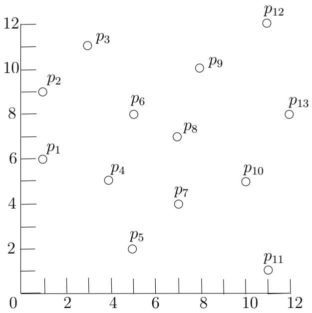
$$dist(p, q) = \min_{p', q' \in P,\ p' \neq q'} dist(p', q').$$

where $dist(.,.)$ represents the Euclidean distance of two points.

Let P be a set of points $\mathbb{R}^d$ and r a real value. The objective of the **close pairs problem** is to output all pairs of distinct points $p, q \in P$ satisfying:

$$dist(p, q) \leq r.$$

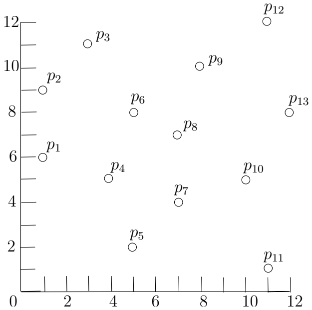
Example: Closest Pair



The answer is (p_6, p_8).

Example: Close Pairs

Assume $r = 4\sqrt{2}$.



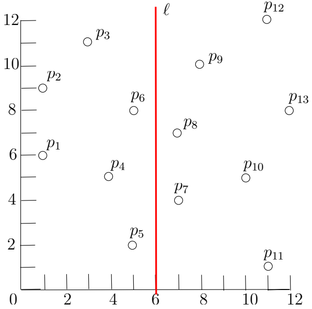
The answer is $\{(p_1, p_4), (p_1, p_2), (p_2, p_3), (p_2, p_6), (p_2, p_4), ...\}$.

Both problems can be easily solved in $O(n^2)$ time where $n = |P|$. We will settle the closest pair problem in $O(n \log n)$ expected time and the close pair problem in $O(n + k)$ expected time, where k is the number of pairs reported.

Closest Pair in 2D

We will focus on 2D.

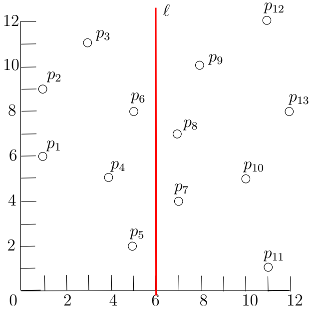
Divide P evenly using a vertical line ℓ. Let P_1 (or P_2) be the set of points on the left (or right) of ℓ. Recursively find the closest pairs in P_1 and P_2, respectively.



The closest pair of P_1 is (p_2, p_3) and that of P_2 is (p_7, p_8).

Closest Pair in 2D

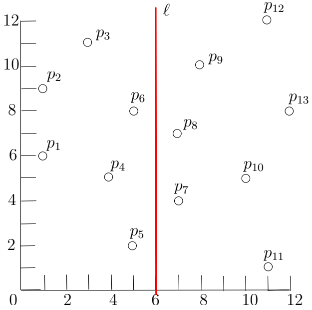
It remains to find the closest pair (p_1, p_2) satisfying $p_1 \in P_1$ and $p_2 \in P_2$ (i.e., p_1, p_2 come from different sides). Call it the **crossing** closest pair.



The crossing closest pair is (p_6, p_8). The global closest pair must be among the two "local" pairs (p_2, p_3), (p_7, p_8), and the crossing pair (p_6, p_8).

Closest Pair in 2D

We now explain how to find the crossing closest pair. Let r_1 be the distance of the closest pair in P_1 and r_2 be the distance of the closest pair in P_2. Define $r = \min\{r_1, r_2\}$.

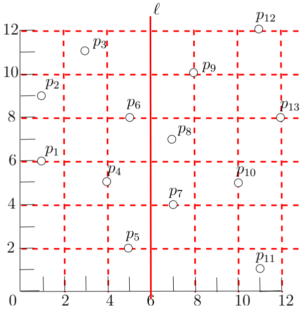


In the above example, $r_1 = \sqrt{8}$, $r_2 = 3$, and $r = \min\{r_1, r_2\} = \sqrt{8}$.

Observation: We care about the crossing closest pair only if its distance is smaller than r.

Closest Pair in 2D

Impose a grid G where (i) each cell is an axis-parallel square with side length $r/\sqrt{2}$, and (ii) ℓ is a line in the grid.



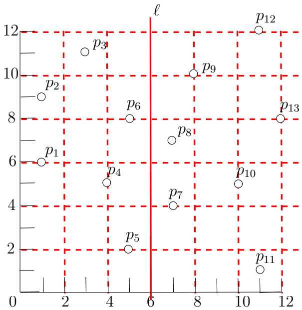
Each point p can be covered by at most 4 cells.

Closest Pair in 2D

For each cell c, denote by $c(P)$ the set of points in P covered by c.

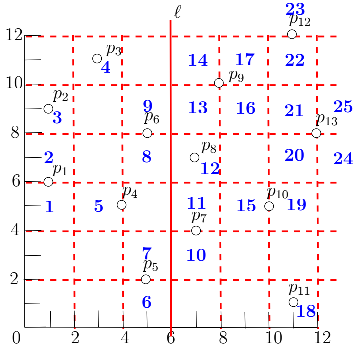
Observation: For every c, $|c(P)| \leq 2 = O(1)$!

Proof: The diagonal of c has length r. Convince yourself that c covering more than 2 points would contradict the definition of r. □



Closest Pair in 2D

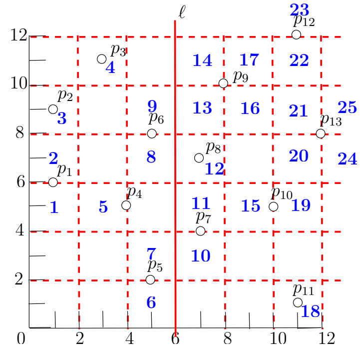
Group the points by the cells they belong. A cell is **non-empty** if it covers at least one point. There can be at most $4n$ non-empty cells.



In the above example, there are 25 non-empty cells.

Closest Pair in 2D

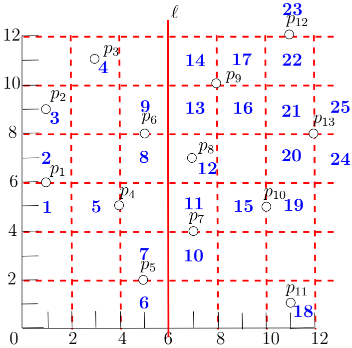
Each cell can be uniquely identified by its centroid's coordinates, which we refer to as the cell's **id**. For each cell c, we create a linked list containing all the points in $c(P)$ (i.e., the set of points covered by c). This can be done using hashing in $O(n)$ expected time.



Closest Pair in 2D

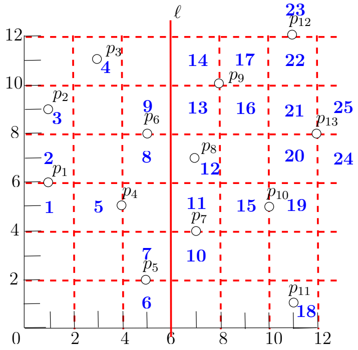
Let c_1, c_2 be two non-empty cells. We say that c_1 is an ***r*-neighbor** of c_2 (and vice versa) if their mindist is at most r.

To find a crossing closest pair within distance r, it suffices to consider non-empty cells c_1, c_2 satisfying (i) c_1 is on the left of ℓ, and c_2 is on the right, and (ii) c_1 and c_2 are r-neighbors.



For example, we need to consider the cell pair (5, 11), but not (5, 15).

Observation: Each non-empty cell c on the left of ℓ has $O(1)$ r-neighbor cells on the right of ℓ.



For example, for Cell 8, we need to consider 8 pairs: (8, 10), (8, 11), (8, 12), (8, 13), (8, 14), (8, 15), (8, 16), (8, 17).

Closest Pair in 2D

The above discussion motivates the following algorithm for finding a crossing closest pair within distance r:

1. **for** every non-empty cell c_1 on the left of ℓ
2. **for** every r-neighbor cell c_2 of c_1 on the right of ℓ
3. calculate the distance of each pair of points $(p_1, p_2) \in c_1(P) \times c_2(P)$
4. **return** the closest one among all the pairs inspected at Line 3, if the pair has distance at most r.

As mentioned, for each c_1, there are $O(1)$ cells c_2 to consider. Since $c_1(P)$ and $c_2(P)$ each contain at most 2 points, each execution of Line 3 takes only $O(1)$ time. The overall algorithm takes $O(n)$ expected time in total.

Think: How to find the cells c_2 for each c_1 in $O(1)$ expected time?

Closest Pair in 2D: Analysis

Let $f(n)$ be the expected running time of our algorithm, it follows that

$$f(n) \leq 2 \cdot f(n/2) + O(n)$$

while $f(n) = O(1)$ for $n \leq 2$.

The recurrence solves to $f(n) = O(n \log n)$.

In the closest-pair problem, we utilized the property that each cell in the grid has $O(1)$ r-neighbor cells.

We now proceed to tackle the close-pairs problem by using the same property. Recall that our objective is to achieve $O(n + k)$ expected time, where k is the number of pairs reported.

Recall the definition of the close-pairs problem.

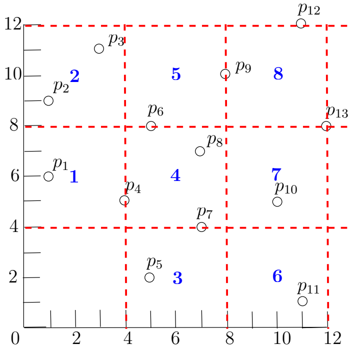
Let P be a set of distinct points $\mathbb{R}^d$ and r a real value. The objective is to output all pairs of distinct points $p, q \in P$ satisfying:

$$dist(p, q) \quad \leq \quad r.$$

We will again focus on 2D space.

Close Pairs in 2D

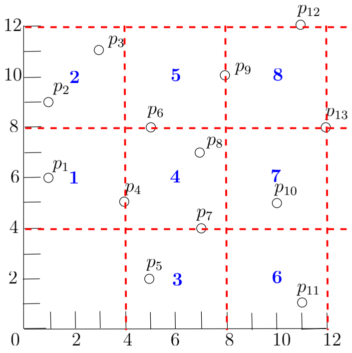
We will explain the algorithm using the same dataset and $r = 4\sqrt{2}$.



Step 1: Impose an arbitrary grid where each square cell has side length $r/\sqrt{2} = 4$. Identify all the non-empty cells.

Close Pairs in 2D

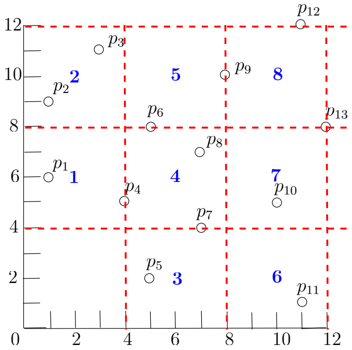
Step 2: For each cell c, let $c(P)$ be the set of points covered by c. Simply report all pairs of distinct points in $c(P)$ — notice that any two points in the same cell must have distance at most r.



For example, 1 pair is reported for Cell 1, and 3 pairs for Cell 8.

Close Pairs in 2D

Step 3: For each cell c_1, identify all of its r-neighbor cells c_2. For every c_2, inspect all pairs of distinct points $(p_1, p_2) \in c_1(P) \times c_2(P)$, and report the ones within distance at most r.



For example, from Cells 2 and 4, inspect all the 8 pairs in $\{p_2, p_3\} \times \{p_4, p_6, p_7, p_8\}$, and report $(p_2, p_4), (p_2, p_6), (p_3, p_6)$.

Close Pairs in 2D: Analysis

Next, we will prove that our algorithm runs in $O(n+k)$ expected time. At first glance, this may look surprising. Recall that in Step 3, for each pair of r-neighbor cells (c_1, c_2), we spend a quadratic amount of time $O(|c_1(P)||c_2(P)|)$, but risk finding no answer pairs at all. Indeed, the core of the analysis is to show that the total time of doing so is bounded by $O(n+k)$.

We will focus on Steps 2 and 3 because Step 1 obviously takes $O(n)$ expected time (hashing).

Close Pairs in 2D: Analysis (Step 2)

Let $c_1, c_2, ..., c_m$ be the non-empty cells, for some $m \geq 1$. Define $n_i = |c_i(P)|$, namely, the number of points covered by c_i, for each $i \in [1, m]$. Clearly $\sum_{i=1}^{m} n_i \geq n$.

The cost of Step 2 is

$$\sum_{i=1}^{m} O(n_i^2)$$

Notice that

$$k \geq \sum_{i=1}^{m} n_i(n_i - 1)/2 = \left(\frac{1}{2}\sum_{i=1}^{m} n_i^2\right) - \left(\frac{1}{2}\sum_{i=1}^{m} n_i\right).$$

We thus have

$$\sum_{i=1}^{m} O(n_i^2) = O(n + k).$$

Close Pairs in 2D: Analysis (Step 3)

We will prove that the cost of Step 3 is $\sum_{i=1}^{m} O(n_i^2)$, and therefore, bounded by $O(n+k)$.

Let c_i and c_j be a pair of r-neighbor cells. Step 3 spends $O(n_i \cdot n_j)$ time to process $c_i(P) \times c_j(P)$. Clearly:

$$n_i \cdot n_j \leq (n_i^2 + n_j^2)/2.$$

Close Pairs in 2D: Analysis (Step 3)

The total cost of Step 3 can be written as

$$O\left(\sum_{i=1}^{m} \sum_{j:\ c_j \text{ is an } r\text{-neighbor of } c_i} (n_i^2 + n_j^2)\right)$$

which is bounded by $O(\sum_{i=1}^{m} n_i^2)$ because a cell has $O(1)$ r-neighbors.

We now conclude that the running time of our close-pairs algorithm is $O(n+k)$ expected.