Connected Components and Dijkstra's Algorithm

CSCI 2100 Teaching Team, Fall 2021

Outline

Today's tutorial covers:

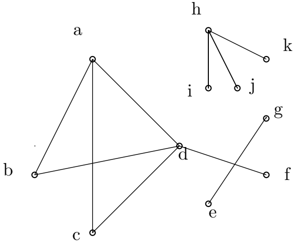
- find connected components using BFS and DFS.
- an example of Dijkstra's Algorithm.

Connected Components

Problem: Let $G = (V, E)$ be an undirected graph. Our goal is to compute all the connected components (CC) of G.

A CC of G includes a set $S \subseteq V$ of vertices such that:

- (Connectivity) any two vertices in S are reachable from each other;
- (Maximality) not possible to add another vertex to S while still satisfying the above requirement.



G

Output:
$\{a, b, c, d, f\}, \{g, e\}, \{h, i, j, k\}$

A Lemma on CCs

Lemma: Take an arbitrary vertex s. The CC of s is the set S of vertices in G reachable from s.

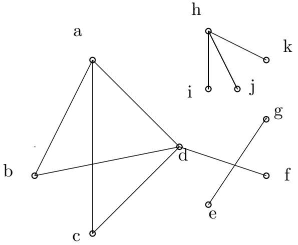
Proof:

- Connectivity: any two vertices in S can reach each other via s.
- Maximality: any vertex outside S is unreachable from s.

□

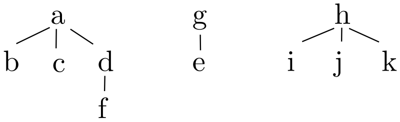
A BFS Solution

1. Run BFS on G with a white source vertex
2. Output the vertex set of the BFS-tree
3. If there is still a white vertex in G, repeat from 1



G

BFS-forest



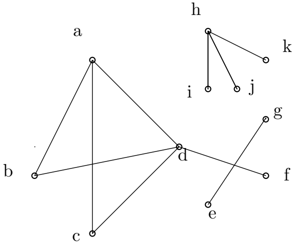
Proof of Correctness

Claim: The vertex set S of every BFS-tree is a CC of G.

Proof: Follows immediately because BFS finds all the vertices reachable from s. $\square$

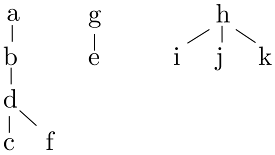
A DFS Solution

1. Run DFS on *G* with a white source vertex
2. Output the vertex set of the DFS-tree
3. If there is still a white vertex in G repeat from 1



G

DFS-forest



Proof of correctness

Claim: The vertex set S of each DFS-tree is a CC of G.

Proof: Let s be the source vertex of DFS. We will show that the DFS-tree contains all and only the vertices reachable from s.

Let v be a vertex reachable from s. At the beginning of DFS, there is a white path from s to v. By the white path theorem, v must be in the subtree of s, namely, in the DFS-tree.

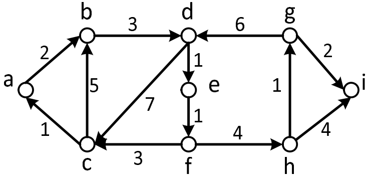
It is obvious that every vertex in the DFS-tree is reachable from s. □

Dijkstra's Algorithm

The algorithm solves the single-source shortest-paths (SSSP) problem on a directed graph $G = (V, E)$ with positive edge weights.

Example

Suppose that the source vertex is a.

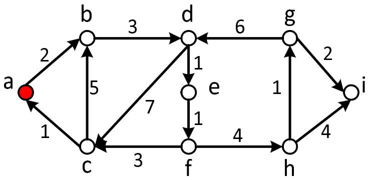


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	∞	nil
c	∞	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$F = \emptyset$ and
$P = \{a, b, c, d, e, f, g, h, i\}$.
Since $dist(a)$ is the smallest among those of vertices in P, pick a.

Example

Relax the out-going edges of a:

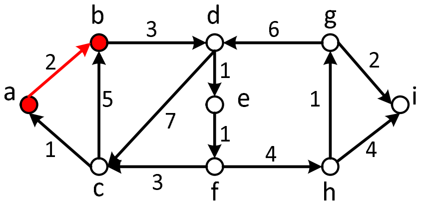


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	$\infty \to 2$	nil $\to a$
c	∞	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$F = \{a\}$ (vertices finalized) and
$P = \{b, c, d, e, f, g, h, i\}$.
Relaxing the edge (a, b).

Example

Relax the out-going edges of b:

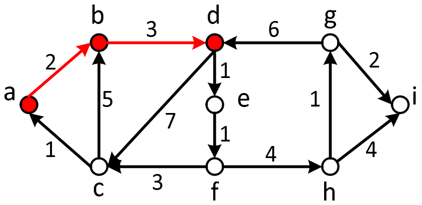


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	∞	nil
d	$\infty \rightarrow 5$	nil $\rightarrow b$
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$F = \{a, b\}$ and
$P = \{c, d, e, f, g, h, i\}$.
Pick b and relax (b, d).

Example

Relax the out-going edges of d:

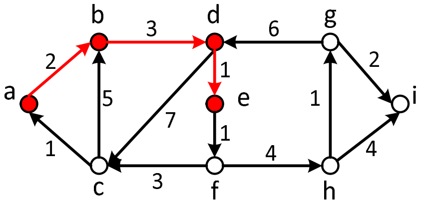


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	$\infty \to 12$	nil $\to d$
d	5	b
e	$\infty \to 6$	nil $\to d$
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$F = \{a, b, d\}$ and
$P = \{c, e, f, g, h, i\}$.
Pick d and relax (d, c) and (d, e).

Example

Relax the out-going edges of e:

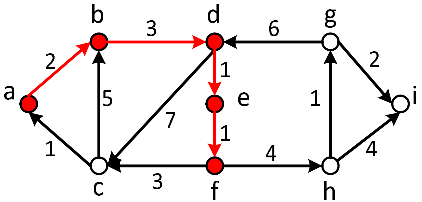


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	12	d
d	5	b
e	6	d
f	$\infty \to 7$	nil $\to e$
g	∞	nil
h	∞	nil
i	∞	nil

$F = \{a, b, d, e\}$ and
$P = \{c, f, g, h, i\}$.

Example

Relax the out-going edges of f:

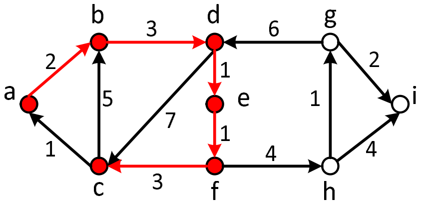


$F = \{a, b, d, e, f\}$ and
$P = \{c, g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	$12 \to 10$	$d \to f$
d	5	b
e	6	d
f	7	e
g	∞	nil
h	$\infty \to 11$	nil $\to f$
i	∞	nil

Example

Relax the out-going edges of c:

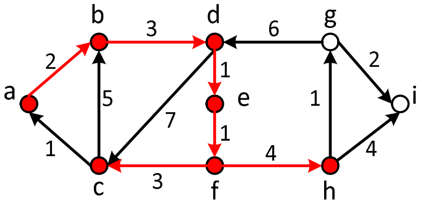


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	∞	nil
h	11	f
i	∞	nil

$F = \{a, b, c, d, e, f\}$ and
$P = \{g, h, i\}$.

Example

Relax the out-going edges of h:

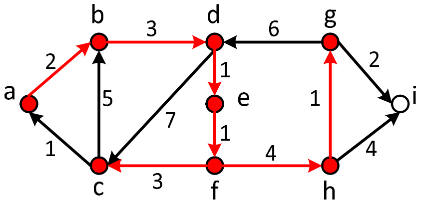


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	$\infty \to 12$	nil $\to h$
h	11	f
i	$\infty \to 15$	nil $\to h$

$F = \{a, b, c, d, e, f, h\}$ and
$P = \{g, i\}$.

Example

Relax the out-going edges of g:

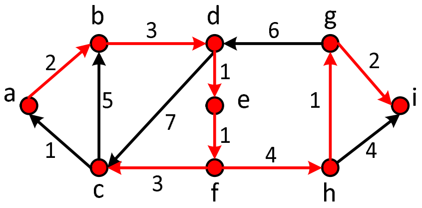


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	$15 \rightarrow 14$	$h \rightarrow g$

$F = \{a, b, c, d, e, f, g, h\}$ and
$P = \{i\}$.

Example

Relax the out-going edges of i:



vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	14	g

$F = \{a, b, c, d, e, f, g, h, i\}$ and
$P = \{\}$.
Done.