# BFS, DFS, and the Proof of White Path Theorem

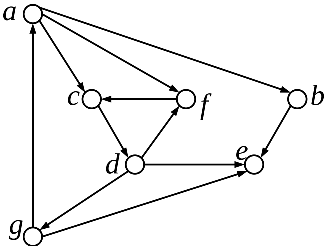
CSCI2100 Tutorial 11

## Introduction

In this tutorial, we will first demonstrate BFS and DFS using examples, and then prove the white path theorem.

Let's first go over the BFS algorithm through a running example on a directed graph.

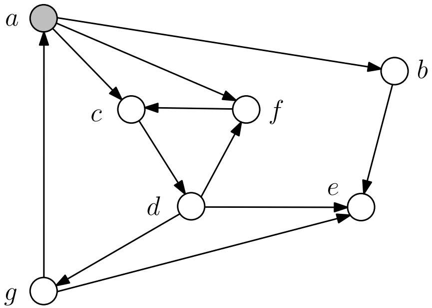
Input




Suppose we start from the vertex $a$, namely $a$ is the root of BFS tree.

## BFS

Firstly, set all the vertices to be white. Then, create a queue $Q$, en-queue the starting vertex *a* and color it gray. Create a BFS Tree with *a* as the root.

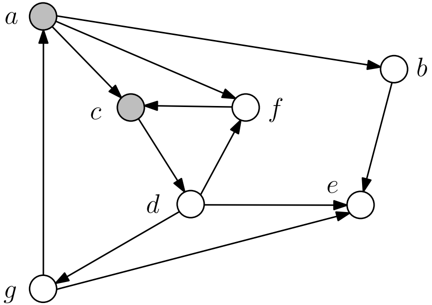



BFS Tree

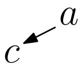
$a$

$Q =$ $a$

BFS

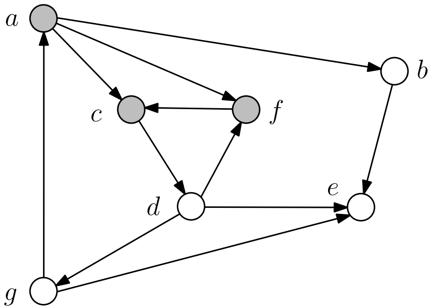



BFS Tree

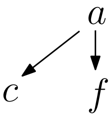



$Q = \quad a \ c$

## BFS

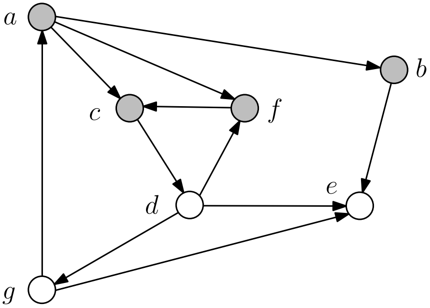



BFS Tree

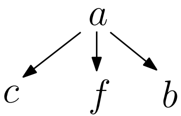



$Q =$ $a\ c\ f$

## BFS

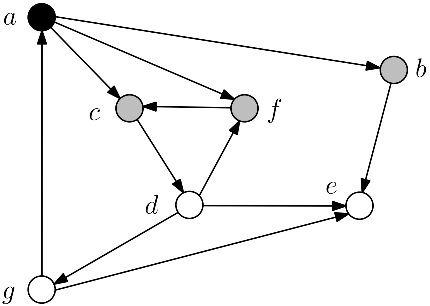



BFS Tree

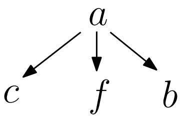



$Q = \underline{\overline{\; a \; c \; f \; b \qquad\qquad\qquad}}$

BFS

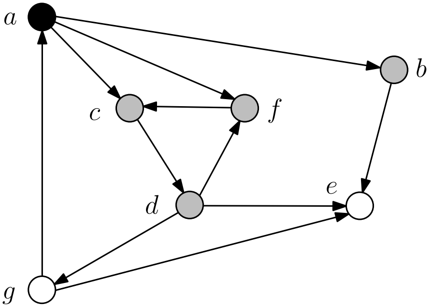



BFS Tree

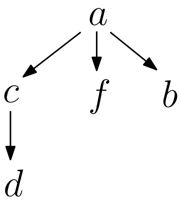



$Q = $ $c \; f \; b$

## BFS

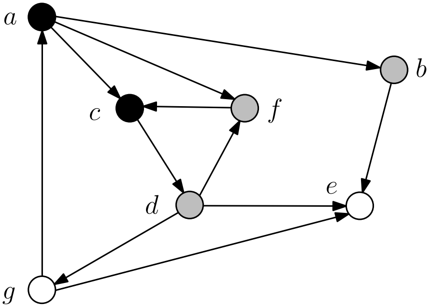



BFS Tree

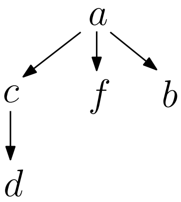



$Q = \underline{\overline{\;c\;\; f\;\; b\;\; d\qquad\qquad\quad}}$

## BFS

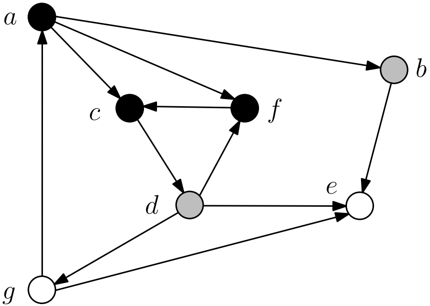


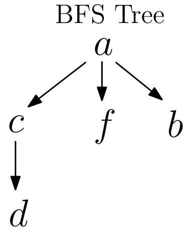

BFS Tree




$$Q = \ f \ \ b \ \ d$$

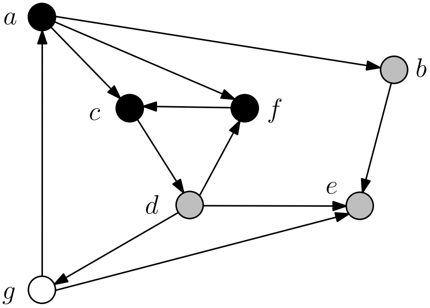
BFS



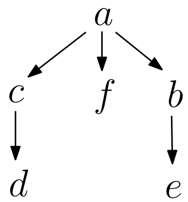




$Q = \underline{\overline{b \ d \qquad\qquad\qquad}}$

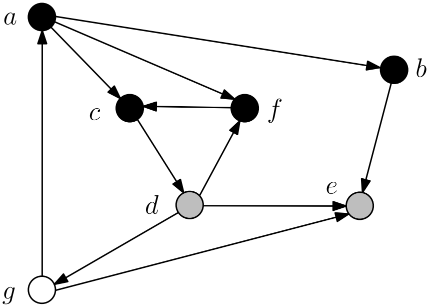
BFS



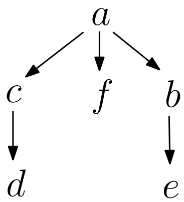

$Q = \underline{\overline{\ b \ d \ e \qquad\qquad\qquad}}$



## BFS

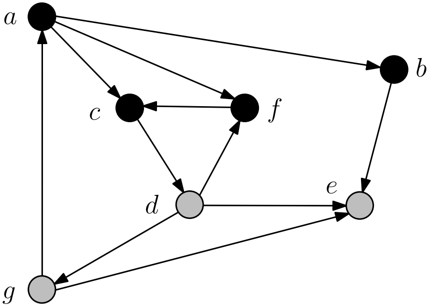



BFS Tree



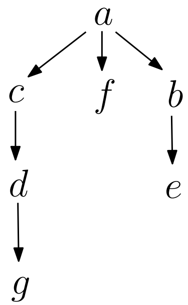

$Q =$ $d$ $e$

## BFS

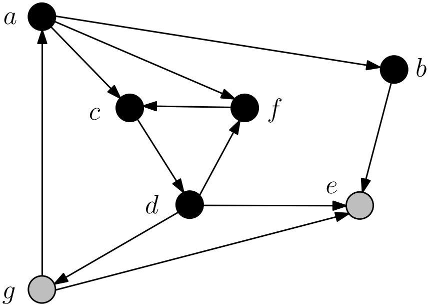


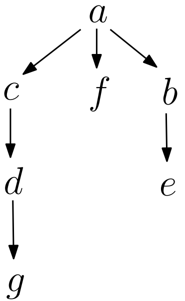

$Q = \; d \; e \; g$

BFS Tree



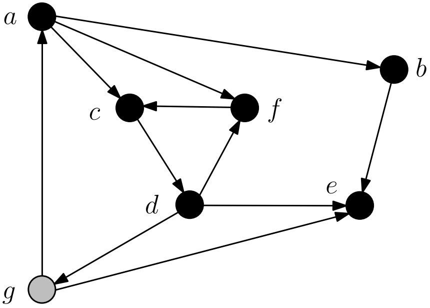
## BFS



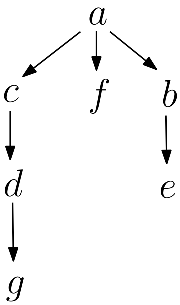




$Q =$ $e\ g$

BFS

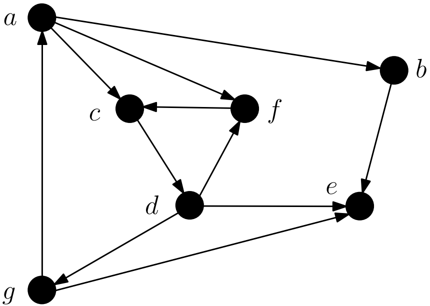



BFS Tree

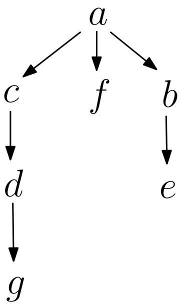



$Q = \ g$

BFS




BFS Tree




$Q =$

Q is empty, algorithm terminated.

## Single Source Shortest Path (SSSP) with Unit Weights

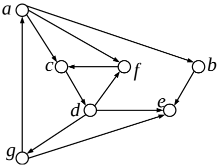
### Input

A directed graph G=$(V, E)$. A vertex s in V as the starting point.

### Goal

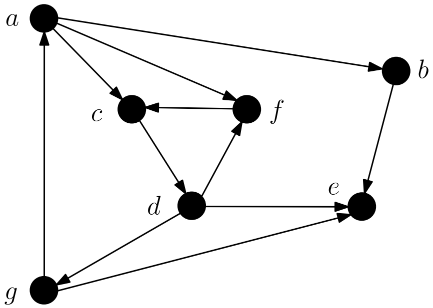
To find, for every other vertex t $\in V \setminus \{s\}$, a shortest path from $s$ to $t$, unless $t$ is unreachable from $s$.

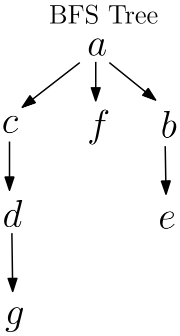
### Example




$a$ is assigned as the starting point.

First step: Do BFS on G using a as the starting point

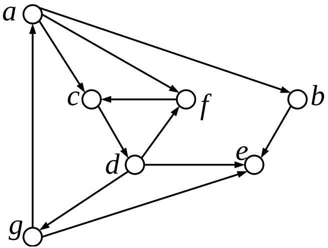



Follow the BFS Tree generated by the *BFS* algorithm, we can find the shortest paths required.

Let's first go over the DFS algorithm through a running example on a directed graph.

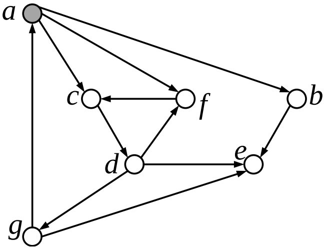
Input




Suppose we start from the vertex $a$, namely $a$ is the root of DFS tree.

## DFS

Firstly, set all the vertices to be white. Then, create a stack $S$, push the starting vertex $a$ into $S$ and color it gray. Create a DFS Tree with $a$ as the root. We also maintain the time interval $I(u)$ of each vertex $u$.

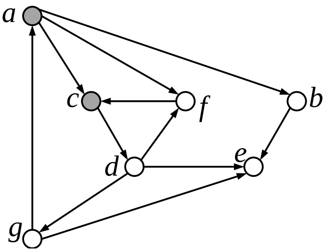



| DFS Tree | Time Interval |
|---|---|
| $a$ | $I(a) = [1, \ ]$ |

$S = (a)$.

DFS

Top of stack: $a$, which has white out-neighbors $b$, $c$, $f$. Suppose we access $c$ first. Push $c$ into $S$.




DFS Tree

$a$
|
$c$

Time Interval

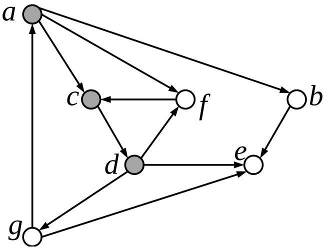
$I(a) = [1, \ ]$

$I(c) = [2, \ ]$

$S = (a, c)$.

## DFS

After pushing $d$ into $S$:

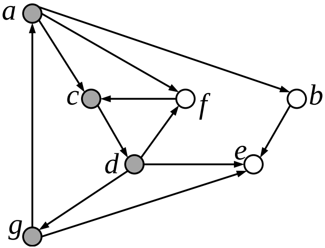



| DFS Tree | Time Interval |
|---|---|
| $a$ | $I(a) = [1, \ ]$ |
| $c$ | $I(c) = [2, \ ]$ |
| $d$ | $I(d) = [3, \ ]$ |

$S = (a, c, d)$.

## DFS

Now $d$ tops the stack. It has white out-neighbors $e$, $f$ and $g$. Suppose we visit $g$ first. Push $g$ into $S$.

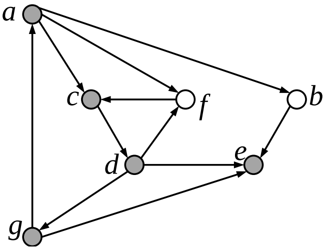



| DFS Tree | Time Interval |
|---|---|
| $a$ | $I(a) = [1, \ ]$ |
| $c$ | $I(c) = [2, \ ]$ |
| $d$ | $I(d) = [3, \ ]$ |
| $g$ | $I(g) = [4, \ ]$ |

$S = (a, c, d, g)$.

## DFS

After pushing $e$ into $S$:

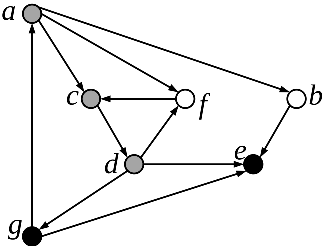



| DFS Tree | Time Interval |
|---|---|
| $a$ | $I(a) = [1, \ ]$ |
| $c$ | $I(c) = [2, \ ]$ |
| $d$ | $I(d) = [3, \ ]$ |
| $g$ | $I(g) = [4, \ ]$ |
| $e$ | $I(e) = [5, \ ]$ |

$S = (a, c, d, g, e)$.

## DFS

$e$ has no white out-neighbors. So pop it from $S$, and color it black. Similarly, $g$ has no white out-neighbors. Pop it from $S$, and color it black.

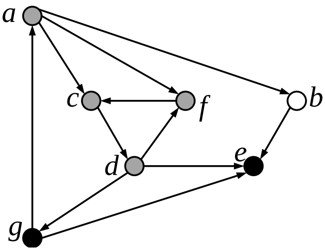



| DFS Tree | Time Interval |
|---|---|
| $a$ | $I(a) = [1, \ ]$ |
| $c$ | $I(c) = [2, \ ]$ |
| $d$ | $I(d) = [3, \ ]$ |
| $g$ | $I(g) = [4, 7]$ |
| $e$ | $I(e) = [5, 6]$ |

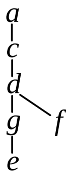
$S = (a, c, d)$.

# DFS

Now $d$ tops the stack again. It still has a white out-neighbor $f$. So, push $f$ into $S$.




DFS Tree



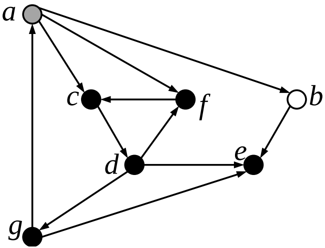

Time Interval

$I(a) = [1, \ ]$
$I(c) = [2, \ ]$
$I(d) = [3, \ ]$
$I(g) = [4, 7]$
$I(e) = [5, 6]$
$I(f) = [8, \ ]$

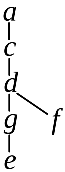
$S = (a, c, d, f)$.

## DFS

After popping $f$, $d$, $c$:




DFS Tree




Time Interval

$I(a) = [1, \ ]$

$I(c) = [2, 11]$

$I(d) = [3, 10]$

$I(g) = [4, 7]$

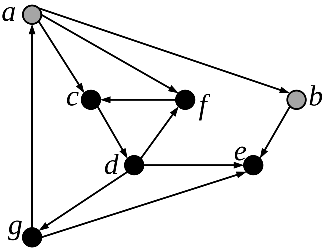
$I(e) = [5, 6]$

$I(f) = [8, 9]$

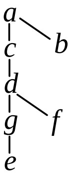
$S = (a)$.

## DFS

Now $a$ tops the stack again. It still has a white out-neighbor $b$. So, push $b$ into $S$.




DFS Tree



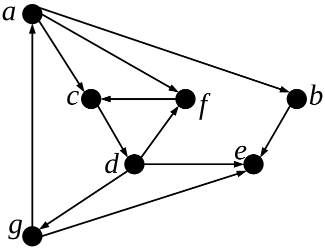

Time Interval

$I(a) = [1, \ ]$
$I(c) = [2, 11]$
$I(d) = [3, 10]$
$I(g) = [4, 7]$
$I(e) = [5, 6]$
$I(f) = [8, 9]$
$I(b) = [12, \ ]$

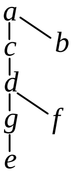
$S = (a, b)$.

## DFS

After popping $b$ and $a$:




DFS Tree




Time Interval

$I(a) = [1, 14]$
$I(c) = [2, 11]$
$I(d) = [3, 10]$
$I(g) = [4, 7]$
$I(e) = [5, 6]$
$I(f) = [8, 9]$
$I(b) = [12, 13]$

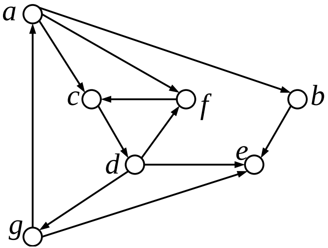
$S = ()$.

Now, there is no white vertex remaining, our algorithm terminates.

## Cycle Detection

### Problem Input:

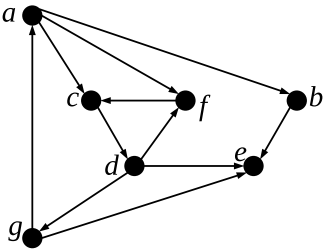
A directed graph.



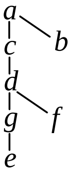

### Problem Output:

A boolean indicating whether the graph contains a cycle.

## First Step: DFS




DFS Tree

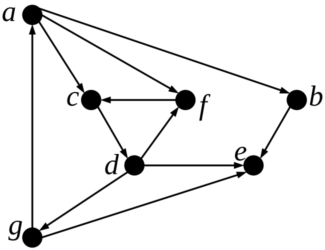



Time Interval

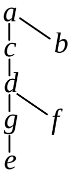
$I(a) = [1, 14]$
$I(c) = [2, 11]$
$I(d) = [3, 10]$
$I(g) = [4, 7]$
$I(e) = [5, 6]$
$I(f) = [8, 9]$
$I(b) = [12, 13]$

**Cycle Theorem:** Let T be an arbitrary DFS-forest of graph G. G contains a cycle if and only if there is a back edge with respect to T.

## Second Step: Try to Find Back Edge




DFS Tree




Time Interval

$I(a) = [1, 14]$

$I(c) = [2, 11]$

$I(d) = [3, 10]$

$I(g) = [4, 7]$

$I(e) = [5, 6]$

$I(f) = [8, 9]$

$I(b) = [12, 13]$

**Parenthesis Theorem:** If $u$ is a proper descendant of $v$ in a DFS-tree of T, then $I(u)$ is contained in $I(v)$.

We proved the cycle theorem in the lecture. Recall that our proof relies on another theorem called the **white path theorem**, which we will establish in the rest of the tutorial.

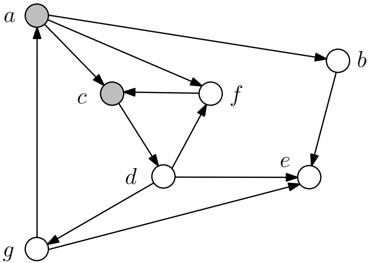
## Proof of White Path Theorem

Recall:

**White Path Theorem**: Let $u$ be a vertex in $G$. Consider the moment when $u$ is pushed into the stack in the DFS algorithm. Then, a vertex $v$ becomes a proper descendant of $u$ in the DFS-forest if and only if the following is true:

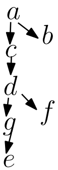
- We can go from $u$ to $v$ by travelling only on white vertices.

## Example




DFS Tree

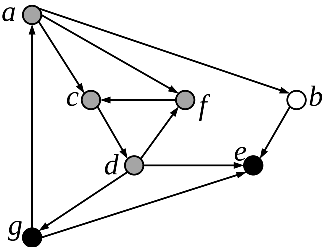

Final DFS Tree



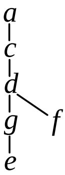

$S = \boxed{a \ \ c}$

**Lemma:** Consider any vertex $u$ in a DFS-tree. If a node $x$ enters the stack while $u$ is in the stack, then $x$ is a descendant of $u$ in a DFS-tree.

The proof is left to you.




DFS Tree




Time Interval

$I(a) = [1, \ ]$

$I(c) = [2, \ ]$

$I(d) = [3, \ ]$

$I(g) = [4, 7]$

$I(e) = [5, 6]$

$I(f) = [8, \ ]$

$S = (a, c, d, f)$.

## Proof of White Path Theorem

**White Path Theorem**: Let $u$ be a vertex in $G$. Consider the moment when $u$ is pushed into the stack in the DFS algorithm. Then, a vertex $v$ becomes a proper descendant of $u$ in the DFS-forest if and only if the following is true:

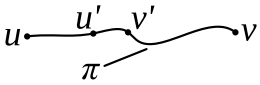
- We can go from $u$ to $v$ by traveling on only white vertices.

**Proof**: The "only-if direction" ($\Rightarrow$): Let $v$ be a descendant of $u$ in the DFS tree. Let $\pi$ be the path from $u$ to $v$ in the tree. By the lemma on Slide 37, all the nodes on $\pi$ entered the stack after $u$. Hence, $\pi$ must be white at the moment when $u$ enters the stack.

## Proof of White Path Theorem

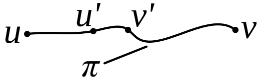
The "if direction" ($\Leftarrow$): When $u$ enters the stack, there is a white path $\pi$ from $u$ to $v$. We will prove that all the vertices on $\pi$ must be descendants of $u$ in the DFS-forest.

Suppose that this is not true. Let $v'$ be the first vertex on $\pi$ — in the order from $u$ to $v$ — that is not a descendant of $u$ in the DFS-forest. Clearly $v' \neq u$. Let $u'$ be the vertex that precedes $v'$ on $\pi$; note that $u'$ is a descendant of $u$ in the DFS-forest.




By the lemma on Slide 37, $u'$ entered the stack after $u$.

## Proof of White Path Theorem




Consider the moment when $u'$ turns black (i.e., $u'$ leaving the stack). Node $u$ must remain in the stack currently (first in last out).

1. The color of $v'$ cannot be white.

   Otherwise, $v'$ is a white out-neighbor of $u$, which contradicts the fact that $u'$ is turning black.

2. Hence, the color of $v'$ must be gray or black.

   Recall that when $u$ entered stack, $v'$ was white. Therefore, $v'$ must have been pushed into the stack while $u$ was still in the stack. By the lemma on Slide 37, $v'$ must be a descendant of $u$. This, however, contradicts the definition of $v'$.