Parallel Acyclic Joins with Canonical Edge Covers

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Abstract

In PODS'21, Hu presented an algorithm in the massively parallel computation (MPC) model that processes any acyclic join with an asymptotically optimal load. In this paper, we present an alternative analysis of her algorithm. The novelty of our analysis is in the revelation of a new mathematical structure — which we name *canonical edge cover* — for acyclic hypergraphs. We prove non-trivial properties for canonical edge covers that offer us a graph-theoretic perspective about why Hu's algorithm works.

Keywords: Joins, Conjunctive Queries, MPC Algorithms, Parallel Computing.

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1 Introduction

Massively parallel join processing has attracted considerable attention in recent years. This line of research makes two types of contributions. The first consists of algorithms that promise excellent performance. The second, more subtle, type of contributions comprises knowledge revealing *mathematical structures* in the underlying problems. The latter is a necessary side-product of the former. In general, as human beings switch to a more generic setting, their knowledge from restrictive settings often proves insufficient, which then necessitates deeper investigation into the problem characteristics. Traditional studies have focused on joins in the RAM computation model [4,14,16,18, 19], a degenerated "parallel" setup having only one machine. Designing algorithms to work with any number of machines poses serious challenges and demands novel findings [3, 7, 8, 10, 12, 13, 15, 20, 21] beyond the RAM literature.

This paper will focus on *acyclic joins*, a class of joins with profound importance in database systems [1, 7–9, 11, 22]. Recently, Hu [8] developed a worst-case optimal massively parallel algorithm for acyclic joins. In the current work, we will provide an alternative, hopefully more accessible, analysis of her elegant algorithm. The real excitement from our analysis is the identification of a new mathematical structure — we call "canonical edge cover" — for acyclic hypergraphs. The structure reveals a unique characteristic of acyclic joins and is a core reason why Hu's algorithm works.

1.1 Problem Definition

Acyclic Joins. Let **att** be a set where each element is called an *attribute*. Let **dom** be another set where each element is called a *value*. We assume a total order on **dom**; if not, manually impose one by ordering the values arbitrarily.

A *tuple* over a set $U \subseteq \mathbf{att}$ is a function $\boldsymbol{u} : U \to \mathbf{dom}$. For each attribute $X \in U$, we refer to $\boldsymbol{u}(X)$ as the *value* of $\boldsymbol{u}$ on X. Given a subset $U' \subseteq U$, define $\boldsymbol{u}[U']$ — the *projection* of $\boldsymbol{u}$ on U' — as the tuple $\boldsymbol{u'}$ over U' such that $\boldsymbol{u'}(X) = \boldsymbol{u}(X)$ for every $X \in U'$. A *relation* is a set R of tuples over the same set U of attributes. We call U the *scheme* of R, a fact denoted as $scheme(R) = U$. If U is the empty set $\emptyset$, then R is also $\emptyset$.

We represent a *join query* (henceforth, simply a "join" or "query") as a set Q of relations. Define $attset(Q) = \bigcup_{R \in Q} scheme(R)$. The query result — denoted as $Join(Q)$ — is the following relation over $attset(Q)$

$$Join(Q) = \left\{ \text{tuple } \boldsymbol{u} \text{ over } attset(Q) \;\middle|\; \forall R \in Q,\, \boldsymbol{u}[scheme(R)] \in R \right\}.$$

If the relations in Q are $R_1, R_2, ..., R_{|Q|}$, we may represent $Join(Q)$ also as $R_1 \bowtie R_2 \bowtie ... \bowtie R_{|Q|}$.

Q can be characterized by a hypergraph $G = (V, E)$ where each vertex in V is a distinct attribute in $attset(Q)$, and each hyperedge in E is the scheme of a distinct relation in Q. E may contain identical hyperedges because two (or more) relations in Q can have the same scheme. The term "hyper" suggests that a hyperedge can have more than two attributes.

A query is *acyclic* if its hypergraph is acyclic. Specifically, a hypergraph $G = (V, E)$ is *acyclic* if we can create a tree T where

- every node in T stores (and, hence, "corresponds to") a distinct hyperedge in E;
- **(connectedness requirement)** for every attribute $X \in V$, the set S of nodes whose corresponding hyperedges contain X forms a connected subtree in T.

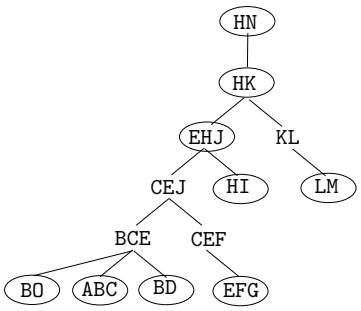


Figure 1: A hyperedge tree example

We will call T a *hyperedge tree* of G (also known as the *join tree* of Q in the literature).

Example 1.1. Consider the hypergraph $G = (V, E)$ where $V = \{\mathtt{A}, \mathtt{B}, ..., \mathtt{O}\}$ and $E = \{\mathtt{ABC}, \mathtt{BD}, \mathtt{BO}, \mathtt{EFG}, \mathtt{BCE}, \mathtt{CEF}, \mathtt{CEJ}, \mathtt{HI}, \mathtt{LM}, \mathtt{EHJ}, \mathtt{KL}, \mathtt{HK}, \mathtt{HN}\}$. Figure 1 shows a hyperedge tree T of G. To understand the connectedness requirement, observe the connected subtree formed by the five hyperedges involving $\mathtt{E}$. □

As G and T both contain "vertices" and "edges", for better clarity we will obey several conventions throughout the paper. A vertex in G will always be referred to as an *attribute*, while the term *node* is reserved for the vertices in T. Furthermore, to avoid confusion with hyperedges, we will always refer to an edge in T as a *link*.

We use m to denote the *input size* of Q, defined as $\sum_{R \in Q} |R|$, namely, the total number of tuples in the relations participating in the join.

Computation Model. We assume the *massively parallel computation* (MPC) model which is popular in designing massively parallel algorithms [3, 7, 8, 10, 12, 13, 15, 20, 21]. In this model, we have p machines, each storing $\Theta(m/p)$ tuples from the relations of a query Q initially. An algorithm executes in *rounds*, each having two phases: in the first phase, each machine performs local computation; in the second, the machines exchange messages (every message must have been generated at the end of the first phase). An algorithm must finish in a constant number of rounds, and when it does, every tuple in $Join(Q)$ must reside on at least one machine. The *load of a round* is the largest number of words received by a machine in that round. The *load of an algorithm* is the maximum load of all the rounds. The objective is to design an algorithm with the smallest load.

Math Conventions. The number p of machines is considered to be at most $m^{1-\epsilon}$, for some arbitrarily small constant $\epsilon > 0$. Every value in **dom** can be represented with $O(1)$ words. Our discussion focuses on *data complexities*, namely, we are interested in the influence of m on algorithm performance. For that reason, we assume that the hypergraph G of Q has $O(1)$ vertices. Given an integer $x \geq 1$, the notation $[x]$ represents the set $\{1, 2, ..., x\}$.

1.2 Previous Results

Fractional Edge Coverings and the AGM bound. Consider a query Q (which may or may not be acyclic) with hypergraph $G = (V, E)$. Associate every hyperedge $e \in E$ with a real-valued *weight* w_e, which falls between 0 and 1. Impose a constraint on every attribute $X \in V$: $\sum_{e \in E: X \in e} w_e \geq 1$, i.e., the total weight of all the hyperedges covering X must be at least 1. A set of weights $\{w_e \mid e \in E\}$ fulfilling all the constraints is a *fractional edge covering* of G. If we define $\sum_{e \in E} w_e$ as the *total weight* of the fractional edge covering, the *fractional edge covering number* of G — denoted as ρ —

is the minimum total weight of all possible fractional edge coverings. A fractional edge covering is *optimal* if its total weight equals ρ.

The *AGM bound*, proved by Atserias, Grohe, and Marx [5], states that the size of $Join(Q)$ is always bounded by $O(m^\rho)$; recall that m is the input size of Q. Furthermore, the bound is tight: in the worst case, $|Join(Q)|$ can indeed reach $\Omega(m^\rho)$ [5].

Simplification for Acyclic Queries: Edge Covers. When Q is acyclic, $G = (V, E)$ always admits an optimal fractional edge covering with *integral* weights [8]. Recall that all the weights w_e ($e \in E$) must fall between 0 and 1. Hence, every weight in an optimal fractional edge covering must be either 0 or 1. This pleasant property allows the reader to connect ρ to edge "covers". A subset $S \subseteq E$ is an *edge cover*[1] of G if every attribute of V appears in at least one hyperedge of S. Thus, the value of ρ is simply the minimum size of all edge covers, namely, the smallest number of hyperedges that we must pick to cover all the attributes.

Join Algorithms in RAM. An algorithm able to answer Q using $O(m^\rho)$ time in the RAM model is worst-case optimal. Indeed, as $|Join(Q)|$ can be $\Omega(m^\rho)$, we need $\Theta(m^\rho)$ time just to output $Join(Q)$ in the worst case. Ngo et al. [17] designed the first algorithm that guarantees a running time of $O(m^\rho)$ for all queries. Since then, the community has discovered more algorithms [4, 14, 16, 18, 19] that are all worst-cast optimal (sometimes up to a polylogarithmic factor) but differ in their own features. For an acyclic Q, an algorithm due to Yannakakis [22] achieves a stronger sense of optimality: his algorithm runs in $O(m + |Join(Q)|)$ time, which is clearly the best regardless of $|Join(Q)|$.

Join Algorithms in MPC. Koutris, Beame, and Suciu [15] showed that, in the MPC model, the AGM bound implies a worst-case lower bound of $\Omega(m/p^{1/\rho})$ on the load of any algorithm that answers a query Q, where m is the input size of Q and ρ is the fractional edge covering number of the hypergraph $G = (V, E)$ defined by Q.

The above negative result has motivated considerable research looking for MPC algorithms whose loads are bounded by $O(m/p^{1/\rho})$, ignoring polylogarithmic factors; such algorithms are worst-case optimal. The goal has been realized only on four query classes. The first consists of all the cartesian-product queries (i.e., the relations in Q have disjoint schemes); see [3, 6, 13] for several optimal algorithms on such queries. The second is the so-called Loomis-Whitney join, where E consists of all the $|V|$ possible hyperedges of $|V| - 1$ attributes; see [15] for an optimal algorithm for such joins. The third class includes every query where all the hyperedges in G contain *at most* two attributes; see [12, 13, 21] for the optimal algorithms. The fourth class comprises all the acyclic queries, which were recently solved by Hu [8] optimally. It is worth pointing out that Hu's algorithm subsumes an earlier algorithm of [9] which is worst-case optimal on a subclass of acyclic queries.

Although it still remains elusive what other query classes can be settled with load $O(m/p^{1/\rho})$, now we know that this is *unachievable* for certain queries. In [8], Hu constructed a class of queries for which every algorithm must incur a load of $\omega(m/p^{1/\rho})$ in the worst case. The result of [8] suggests that additional parameters — other than m, p, and ρ — are needed to describe the worst-case optimality of an ideal MPC algorithm. We will not delve into the issue further because it does not apply to acyclic queries (the focus of this paper), but the reader may consult the recent works [8, 20] for the latest development on that issue. Finally, we remark that several algorithms [2, 9, 10] are able to achieve a load sensitive to the join size $|Join(Q)|$.

[1] In case the reader is wondering, the literature uses the words "covering" and "cover" exactly the way they are used in our paper.

1.3 Our Contributions

The first, easy-to-discern, contribution of our paper is a new analysis of Hu's algorithm [8] for acyclic queries. Our second contribution is the introduction of *canonical edge cover* as a mathematical structure inherent in acyclic queries. We prove a suite of graph-theoretic properties for canonical edge covers and use them to give a more fundamental interpretation of the design choices in Hu's algorithm. The rest of the section will provide an overview of our results and techniques.

Clustering, k-Groups, k-Products, and Induced Loads. We first create a conceptual framework to state Hu's and our results on a common ground. Define a *clustering* of E (the hyperedge set of G) as a set $\{E_1, E_2, ..., E_s\}$ for some $s \geq 1$ where (i) each E_i is a subset of E, $i \in [s]$, and (ii) $\bigcup_i E_i = E$. We call each E_i a *cluster*; note that the clusters need *not* be disjoint.

Fix an arbitrary clustering $C = \{E_1, E_2, ..., E_s\}$. Given an integer $k \geq 1$, define a *k-group* of C as a collection of k hyperedges, each taken from a distinct cluster.

Example 1.2. Let $G = (V, E)$ be the hypergraph in Example 1.1 (Figure 1). $C = \{\{$BO, BCE, CEJ$\}$, $\{$ABC, BCE, CEJ$\}$, $\{$BD, BCE, CEJ$\}$, $\{$EFG, CEF, CEJ$\}$, $\{$HI$\}$, $\{$EHJ$\}$, $\{$LM, KL$\}$, $\{$HK$\}$, $\{$HN$\}\}$ is a clustering of E. A 3-group example is $\{$ABC, BD, EFG$\}$. Note that the hyperedges in a k-group need not be distinct. For example, $\{$CEJ, CEJ, CEJ$\}$ is also a 3-group: the first CEJ is taken from the cluster $\{$ABC, BCE, CEJ$\}$, the second from $\{$BD, BCE, CEJ$\}$, and the third from $\{$EFG, CEF, CEJ$\}$. For a non-example, $\{$ABC, LM, KL$\}$ is not a 3-group. □

For each hyperedge $e \in E$, let $R(e)$ represent the relation in Q whose scheme is e. Given a k-group K of the clustering C, we define the *Q-product* of K as $\prod_{e \in K} |R(e)|$ (i.e., the cartesian-product size of all the relevant relations). Given an integer k, we define the *max (k, Q)-product* of C — denoted as $P_k(Q, C)$ — as the maximum Q-product of all the k-groups of C.

Example 1.3. Continuing on the previous example, the Q-product of the 3-group $\{$ABC, BCE, CEJ$\}$ is $|R($ABC$)| \cdot |R($BCE$)| \cdot |R($CEJ$)|$, while that of the 3-group $\{$CEJ, CEJ, CEJ$\}$ is $|R($CEJ$)|^3$. □

Define the *Q-induced load* of C as

$$\max_{k=1}^{s} \left(P_k(Q, C)/p\right)^{1/k} \tag{1}$$

As $P_k(Q, C) \leq m^k$ for any $k \in [s]$, it must hold that (1) $\leq m/p^{1/s}$.

We can now give a more detailed account of Hu's result [8]. She proved that the load of her algorithm is bounded by $O(L)$, where L is the Q-induced load of a *certain* clustering with size $s = \rho$, and ρ is the fractional edge covering number of G. It thus follows immediately that $L \leq m/p^{1/s}$. In [8], Hu presented a recursive procedure to identify the clustering C whose Q-induced load equals the target L. The procedure, however, is somewhat sophisticated, making it difficult to describe the target C in a succinct manner. Such difficulty is unjustified, especially given the algorithm's elegance, and indicates the existence of a hidden mathematical structure.

Our Results and Techniques. A hypergraph G can have many optimal edge covers (all of which must have size ρ). While Hu's analysis [8] assumes an arbitrary optimal edge cover, we will be choosy about what we work with. In Figure 1, the 9 circled nodes constitute a *canonical edge cover* F of G. Let us give an informal but intuitive explanation of how to construct this F. After rooting the tree in Figure 1 at HN, we add to F all the leaf nodes: BO, ABC, BD, EFG, HI, LM. Then, we process the non-leaf nodes bottom up. In processing BCE, we ask: which attributes will disappear as we

ascend further in the tree? The answer is B, which is thus a "disappearing" attribute of BCE. Then, we ask: does F already cover B? The answer is yes, due to the existence of BO; we therefore do *not* include BCE in F. We process BCE, CEF, and CEJ similarly, none of which enters F. At EHJ, we find disappearing attributes E and J. In general, as long as one disappearing attribute has not been covered by F, we pick the node; this is why EHJ is in F. The other nodes HK and HN in F are chosen based on the same reasoning.

We show that a canonical edge cover determined this way has appealing properties which fit the recursive strategy behind Hu's algorithm very well. At a high level, Hu's algorithm works by simplifying G into a number of "residual" hypergraphs to be processed recursively. Interestingly, with trivial modifications (such as removing the attributes that have become irrelevant), a canonical edge cover of G *remains canonical on every residual hypergraph.* This is the most crucial property we utilize to relate the load of the original query to those of the "residual queries" in forming up a working recurrence.

Our techniques also provide a simple and natural way to pinpoint a clustering C that can be used to bound the algorithm's load. Consider the canonical edge cover F shown in Figure 1 (the circled nodes). For each node in F, take a "signature path" by walking up and stopping right before reaching its lowest proper ancestor in F. For example, the signature path of ABC is {ABC, BCE, CEJ} (note: the path does not contain EHJ). Likewise, the signature path of LM is {LM, KL}. The signature paths of all the nodes in F together produce the clustering C given in Example 1.2. Our main result (Theorem 9) states that the Q-induced load of C is an upper bound on the load of Hu's algorithm. Because C has a size at most ρ, the algorithm's load is thus bounded by $O(m/p^{1/\rho})$.

2 Canonical Edge Covers for Acyclic Hypergraphs

This section is purely graph theoretic: we will establish several new properties for acyclic hypergraphs. Let $G = (V, E)$ be an acyclic hypergraph. A hyperedge $e_1 \in E$ is *subsumed* if it is a subset of another hyperedge $e_2 \in E$, i.e., $e_1 \subseteq e_2$. If an attribute X appears in only a single hyperedge, we call X an *exclusive attribute*; otherwise, X is *non-exclusive*. Unless otherwise stated, we allow G to be an arbitrary acyclic hypergraph. In particular, this means that E can contain two or more hyperedges with the same attributes (nonetheless, they are still distinct hyperedges) and may even have empty hyperedges (i.e., with no attributes at all). G is *clean* if E has no subsumed edges. Some of our results will apply only to clean hypergraphs.

Denote by T a hyperedge tree of G (the existence of T is guaranteed; see Section 1.1). By rooting T at an arbitrary leaf, we can regard T as a *rooted* tree. Make all the links[2] of T point *downwards*, i.e., from parent to child. This way, T becomes a directed acyclic graph.

Now that there are two views of T (i.e., undirected and directed), we ought to be careful with terminology. By default, we will treat T as a directed tree. Accordingly, a *leaf* of T is a node with out-degree 0, a *path* is a sequence of nodes where each node has a link pointing to the next node, and a *subtree* rooted at a node e is the directed tree induced by the nodes reachable from e in T. Sometimes, we may revert back to the undirected view of T. In that case, we use the term *raw leaf* for a leaf in the undirected T (i.e., a raw leaf can be a leaf or the root under the directed view)

[2]Remember that we refrain from saying "edges" of T; see Section 1.1.

2.1 Fundamental Definitions and Properties

Summits and Disappearing Attributes. We say that the root of T is the *highest* node in T and, in general, a node is *higher* (or *lower*) than any of its proper descendants (or ancestors). For each attribute $X \in V$, we define the *summit* of X as the highest node (a.k.a. a hyperedge) that contains X. If node e is the summit of X, we call X a *disappearing* attribute in e. By acyclicity's connectedness requirement (Section 1.1), X can appear only in the subtree rooted at e and hence "disappears" as soon as we leave the subtree.

Example 2.1. Let $G = (V, E)$ be the hypergraph in Example 1.1 whose (rooted) hypergraph tree T is shown in Figure 1. The summit of `C` is node `CEJ`. Thus, `C` is a disappearing attribute of `CEJ`. Node `EHJ` is the summit of `E` and `J`. Hence, both `E` and `J` are disappearing attributes of `EHJ`. □

Canonical Edge Cover. We say that a subset $S \subseteq E$ *covers* an attribute $X \in V$ if S has a hyperedge containing X. Recall that an optimal edge cover of G is the smallest S covering every attribute in V. Optimal edge covers are not unique. Some are of particular importance to us; and we will identify them as "canonical". Towards a procedural definition, consider the following algorithm:

edge-cover (T) /* T is rooted */
1. $F_{\text{tmp}} = \emptyset$
2. obtain a reverse topological order $e_1, e_2, ..., e_{|E|}$ of the nodes (i.e., hyperedges) in T
3. **for** $i = 1$ to $|E|$ **do**
4. **if** e_i has a disappearing attribute not covered by F_{tmp} **then** add e_i to F_{tmp}
5. **return** F_{tmp}

Lemma 1. The output of *edge-cover* — denoted as F — is an optimal edge cover of G, and does not depend on the reverse topological order at Line 2. Furthermore, if G is clean, F includes all the raw leaves of T.

All the missing proofs can be found in the appendix. We refer to F as the *canonical edge cover* (CEC) of G induced by T. The size of F is precisely the fractional edge covering number ρ of Q.

Example 2.2. Continuing on the previous example, consider the reverse topological order of T: `ABC`, `BD`, `BO`, `BCE`, `EFG`, `CEF`, `CEJ`, `HI`, `EHJ`, `LM`, `KL`, `HK`, `HN`. When processing `ABC`, edge-cover adds it to F_{tmp} because `ABC` has a disappearing attribute `A` and yet $F_{\text{tmp}} = \emptyset$. When processing `BCE`, $F_{\text{tmp}} = \{\texttt{ABC}, \texttt{BD}, \texttt{BO}\}$. `BCE` has a disappearing attribute `B`, which, however, has been covered by F_{tmp}. Thus, `B` is not added to F_{tmp}. The final output of the algorithm is $F = \{$`ABC`, `BD`, `BO`, `EFG`, `HI`, `LM`, `EHJ`, `HK`, `HN`$\}$, which is the CEC of G induced by T. □

Signature Paths. Whenever F includes the root of T, we can define a *signature path* — denoted as $\text{sigpath}(f, T)$ — for each node (i.e., hyperedge) $f \in F$. Specifically, $\text{sigpath}(f, T)$ is a set of nodes defined as follows:

- If f is the root of T, $\text{sigpath}(f, T) = \{f\}$.
- Otherwise, let $\hat{f}$ be the lowest node in F that is a proper ancestor of f. Then, $\text{sigpath}(f, T)$ is the set of nodes on the path from $\hat{f}$ to f, except $\hat{f}$.

Example 2.3. Consider the set F obtained in the previous example. If $f =$ `HN`, then the signature path of f is $\{$`HN`$\}$. If $f =$ `ABC`, then $\hat{f} =$ `EHJ`; and the signature path of f is $\{$`ABC`, `BCE`, `CEJ`$\}$. □

(Clean G) Clustering, Anchor Leaf, and Anchor Attribute. Consider $G = (V, E)$ now as a clean hypergraph. Let F be the CEC of G induced by a hyperedge tree T of G. As F contains the root and leaves of T (Lemma 1), $\{\text{sigpath}(f, T) \mid f \in F\}$ is a clustering of E. If f is not the root of T, we call $\text{sigpath}(f, T)$ a *non-root* cluster.[3]

Let f° be a leaf node in F, and $\hat{f}$ be the lowest proper ancestor of f° in F. We call f° an *anchor leaf* of T if two conditions are satisfied:

- $\hat{f}$ has no non-leaf proper descendants in F.
- f° has an attribute A° such that $A^\circ \notin \hat{f}$ but $A^\circ \in e$ for every node $e \in \text{sigpath}(f^\circ, T)$.

A° will be referred to as an *anchor attribute* of f°.

Lemma 2. If G is clean, F always contains an anchor leaf.

Example 2.4. From the F constructed earlier, we obtain the clustering $C = \{\{\texttt{BO}, \texttt{BCE}, \texttt{CEJ}\}, \{\texttt{ABC}, \texttt{BCE}, \texttt{CEJ}\}, \{\texttt{BD}, \texttt{BCE}, \texttt{CEJ}\}, \{\texttt{EFG}, \texttt{CEF}, \texttt{CEJ}\}, \{\texttt{HI}\}, \{\texttt{EHJ}\}, \{\texttt{LM}, \texttt{KL}\}, \{\texttt{HK}\}, \{\texttt{HN}\}\}$. Other than $\{\texttt{HN}\}$, all the clusters in C are non-root clusters. `ABC` is an anchor leaf of T with an anchor attribute `C`. `HI` is another anchor leaf with an anchor attribute `I`. For a non-example, `BD` is not an anchor leaf because it does not have an attribute that exists in all the nodes in $\text{sigpath}(\texttt{BD}, T) = \{\texttt{BD}, \texttt{BCE}, \texttt{CEJ}\}$. Furthermore, `LM` is not an anchor leaf because `HK`, the lowest proper ancestor of `LM` in F, has a non-leaf proper descendant in F (i.e., `EHJ`). □

2.2 (Clean G) Properties on Residual Hypergraphs

This subsection assumes $G = (V, E)$ to be clean. Let T be a hyperedge tree of G and F be the CEC induced by T. Fix an arbitrary anchor leaf f° of T and an anchor attribute A° of f°. We will analyze how the CEC changes as G is simplified based on f° and A°.

2.2.1 Simplification 1

The first simplification is based on removing attribute A° from G.

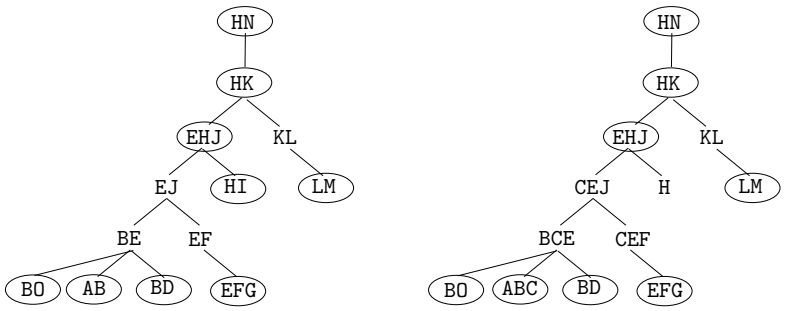
Residual Hypergraph. Let $G' = (V', E')$ be the *residual hypergraph* obtained by eliminating A° from G: $V' = V \setminus \{A^\circ\}$, and E' collects a hyperedge $e' = e \setminus \{A^\circ\}$ for every $e \in E$.[4] We characterize the one-one correspondence between E' and E by introducing a function $map(e) = e'$ and its inverse function $map^{-1}(e') = e$. Let T' be the hyperedge tree of G' obtained by discarding A° from every node in T (note: G' is not necessarily clean).

Canonical Edge Cover. Define

$$F' \quad = \quad \begin{cases} F \setminus \{f^\circ\} & \text{if } map(f^\circ) \text{ is subsumed in } G' \\ \{map(f) \mid f \in F\} & \text{otherwise} \end{cases} \tag{2}$$

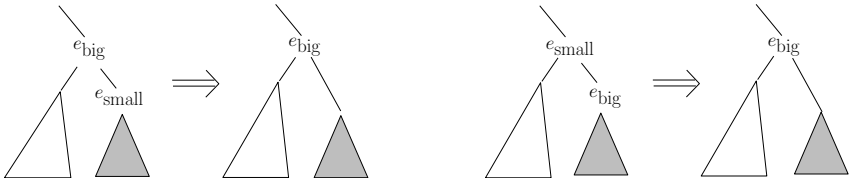
Example 2.5. Continuing on the previous example, if we choose $f^\circ = \texttt{ABC}$ with $A^\circ = \texttt{C}$ and eliminate `C` from the tree T in Figure 1, we obtain the hyperedge tree T' in Figure 2a, where the circled nodes constitute the set F'. Similarly, if we choose $f^\circ = \texttt{HI}$ with $A^\circ = \texttt{I}$, then T' and F' are as demonstrated in Figure 2b. □

[3] If f is the root of T, $\text{sigpath}(f, T)$ contains just f itself.
[4] If $e = \{A^\circ\}$, E' collects $e' = \emptyset$.



(a) T' and F' after removing C (b) T' and F' after removing I

Figure 2: Residual hypergraphs



(a) e_{big} parents e_{small} (b) e_{small} parents e_{big}

Figure 3: Two cases of cleansing

Lemma 3. If G is clean, F' is the CEC of G' induced by T'. Furthermore, if $map(f^\circ)$ is subsumed in G', then A° must be an exclusive attribute in f°.

As a corollary, if $map(f^\circ)$ is subsumed in G', then every hyperedge of G, except f°, is directly retained in G'; furthermore, $map(f^\circ)$ is the only subsumed edge in G'. The next lemma gives another property of F' that holds no matter if G is clean.

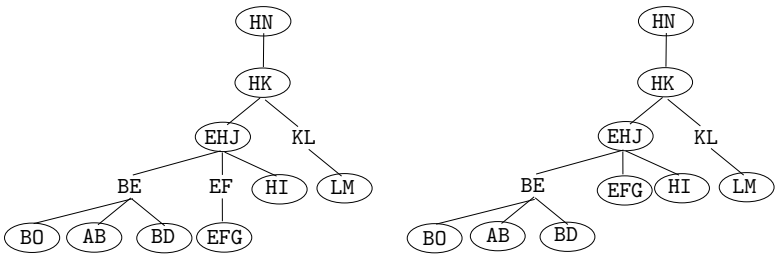
Lemma 4. If a hyperedge e' of G' is subsumed, then $e' \notin F'$

Cleansing. Even though G is clean, the residual hypergraph G' may contain subsumed hyperedges. Next, we describe a *cleansing* procedure which converts G' into a clean hypergraph $G^* = (V', E^*)$ (note that G^* has the same vertices as G') and converts T' into a rooted hyperedge tree T^* of G^*.

Cleansing is simple if $map(f^\circ)$ is subsumed in G'. In this case, G^* is the hypergraph obtained by removing $map(f^\circ)$ from G', and T^* is the tree obtained by removing the leaf $map(f^\circ)$ from T'. If $map(f^\circ)$ is not subsumed, the cleansing algorithm is:

cleanse (G', T') /* condition: $map(f^\circ)$ not subsumed */
1. $G^* = G', T^* = T'$
2. **while** G^* has hyperedges e_{small} and e_{big} such that $e_{\text{small}} \subseteq e_{\text{big}}$ and they are connected by a link in T^* **do**
3. remove e_{small} from G^* and T^*
   /* $e_{\text{small}} \notin F'$ by Lemma 4 */
4. **if** e_{big} was the parent of e_{small} in T^* **then**
5. make e_{big} the new parent for all the child nodes of e_{small}; see Figure 3a
   else
6. make e_{big} the new parent for the child nodes of e_{small}, and
   make e_{big} a child of the (original) parent of e_{small} in T^*; see Figure 3b
7. **return** G^* and T^*

At the end of cleansing, we always set $F^* = F'$, regardless of whether $map(f^\circ)$ is subsumed.



(a) After removing EJ (b) After removing EF

Figure 4: Simplification 1

Lemma 5. After cleansing, F^* is the CEC of G^* induced by T^*.

Example 2.6. In Example 2.5, the residual hypergraph G' in Figure 2a has two subsumed hyperedges EJ and EF, each removed by an iteration of cleanse. Suppose that the first iteration sets $e_{\text{small}} = \texttt{EJ}$ and $e_{\text{big}} = \texttt{EHJ}$ (this is a case of Figure 3a). Figure 4a illustrates the T^* after removing EJ. The next iteration sets $e_{\text{small}} = \texttt{EF}$ and $e_{\text{big}} = \texttt{EFG}$ (a case of Figure 4b). Figure 4b illustrates the T^* after removing EF. In both Figure 4a and 4b, the circled nodes constitute the CEC of G^* induced by T^*. □

Distinct Clusters Lemma. The next property concerns the hypergraph $G^* = (V', E^*)$ after cleansing and the original hypergraph $G = (V, E)$. Recall that T^* and T are hyperedge trees of G^* and G, respectively. Before proceeding, the reader should recall that every hyperedge $e^* \in E^*$ corresponds to a distinct hyperedge $e \in E$, which is the hyperedge given by $map^{-1}(e^*)$.

Consider once again the CEC F of G, i.e., the original hypergraph, induced by T. As mentioned in Section 2.1, $C = \{\text{sigpath}(f, T) \mid f \in F\}$ is a clustering of E. By the same reasoning, because F^* is the CEC of G^* induced by T^* (Lemma 5), $C^* = \{\text{sigpath}(f^*, T^*) \mid f^* \in F^*\}$ must be a clustering of E^*. The following lemma draws a connection between C and C^*:

Lemma 6 (Distinct Clusters Lemma)**.** For any $1 \leq k \leq |F^*|$, if $\{e_1^*, ..., e_k^*\}$ is a k-group of C^*, then $\{map^{-1}(e_1^*), ..., map^{-1}(e_k^*)\}$ is a k-group of C.

By definition of k-group, $e_1^*, ..., e_k^*$ originate from k distinct clusters in C^*. The lemma promises k different clusters in C each containing a distinct hyperedge in $\{map^{-1}(e_1^*), ..., map^{-1}(e_k^*)\}$.

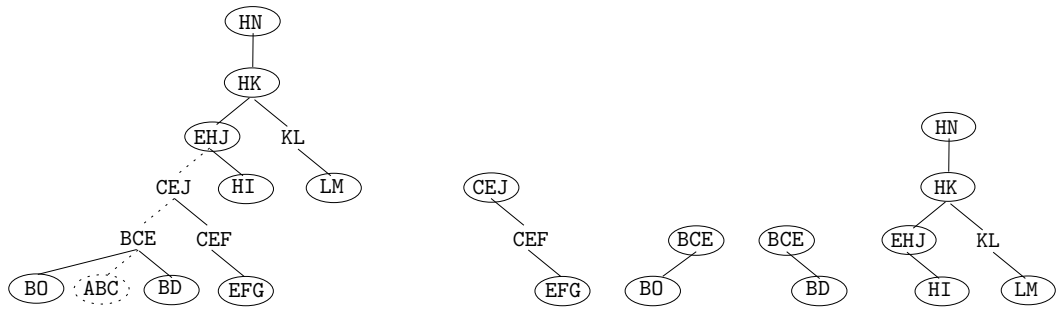
Example 2.7. Consider the T^* (and hence G^*) and F^* illustrated in Figure 4b. The clustering C^* is {{AB, BE}, {BO, BE}, {BD, BE}, {EFG}, {EHJ}, {HI}, {LM, KL}, {HK}, {HN}}. Because {BE, EFG, KL} is a 3-group of C^*, Lemma 6 asserts that $\{map^{-1}(\texttt{BE}), map^{-1}(\texttt{EFG}), map^{-1}(\texttt{KL})\} = \{\texttt{BCE}, \texttt{EFG}, \texttt{KL}\}$ must be a 3-group of the clustering C in Example 2.4. □

2.2.2 Simplification 2

The second simplification decomposes G into multiple hypergraphs based on $\text{sigpath}(f^\circ, T)$.

Decomposition. Define Z to be the set of nodes z in T satisfying: z is not in $\text{sigpath}(f^\circ, T)$ but the parent of z is. For each $z \in Z$, define a rooted tree T_z^* as follows:

- The root of T_z^* is the parent of z in T.
- The root of T_z^* has only one child in T_z^*, which is z.



(a) The dotted path is sigpath(ABC, T) (b) $T^*_{\texttt{CEF}}$ (c) $T^*_{\texttt{BO}}$ (d) $T^*_{\texttt{BD}}$ (e) $\bar{T}^*$

Figure 5: Decomposition

- The subtree rooted at z in T^*_z is the same as the subtree rooted at z in T.

Separately, define $\bar{T}^*$ as the rooted tree obtained by removing from T the subtree rooted at the highest node in sigpath(f°, T).

From each T^*_z, generate a hypergraph $G^*_z = (V^*_z, E^*_z)$. Specifically, E^*_z includes all and only the nodes (each being a hyperedge) in T^*_z, and V^*_z is the set of attributes appearing in at least one hyperedge in E^*_z. Likewise, from $\bar{T}^*$, generate a hypergraph $\bar{G}^* = (\bar{V}^*, \bar{E}^*)$ where $\bar{E}^*$ includes all and only the nodes in $\bar{T}^*$, and $\bar{V}^*$ is the set of attributes appearing in at least one hyperedge in $\bar{E}^*$.

Because G is clean, so must be all the generated hypergraphs. Furthermore, each of them has fewer edges than G.[5] For each $z \in Z$, T^*_z is a hyperedge tree of G^*_z; similarly, $\bar{T}^*$ is a hyperedge tree of $\bar{G}^*$.

Example 2.8. In our running example, f° = ABC, whose signature path is sigpath(f°, T) = {ABC, BCE, CEJ}; see Figure 5a. Z = {BO, BD, CEF}. Figure 5b, 5c, and 5d illustrate T^*_z for z = CEF, BO, and BD, respectively. Figure 5e gives $\bar{T}^*_z$. □

Canonical Edge Covers. Recall that F is the CEC of G induced by T. Next, we derive the CECs of the hypergraphs generated from the decomposition. For each $z \in Z$, define

$$F^*_z \quad = \quad \{\text{parent of } z\} \cup (F \cap E^*_z). \tag{3}$$

Also, define

$$\bar{F}^* \quad = \quad F \cap \bar{E}^*. \tag{4}$$

Lemma 7. For each node $z \in Z$, F^*_z is the CEC of G^*_z induced by T^*_z. Furthermore, $\bar{F}^*$ is the CEC of $\bar{G}^*$ induced by $\bar{T}^*$.

Example 2.9. We have circled the nodes in F^*_z in Figure 5b, 5c, and 5d for z = CEF, BO, and BD, respectively. Similarly, the circled nodes in Figure 5e constitute $\bar{F}^*_z$. □

Distinct Clusters Lemma 2. We close the section with a property resembling Lemma 6.

Consider any $z \in Z$. Because $G^*_z = (V^*_z, E^*_z)$ is clean and F^*_z is the CEC of G^*_z induced by T^*_z, $C^*_z = \{\text{sigpath}(f^*, T^*_z) \mid f^* \in F^*_z\}$ is a clustering of E^*_z. Similarly, regarding $\bar{G}^* = (\bar{V}^*, \bar{E}^*)$, $\bar{C}^* = \{\text{sigpath}(f^*, \bar{T}^*) \mid f^* \in \bar{F}^*\}$ is a clustering of $\bar{E}^*$.

Define a *super-k-group* to be a set of hyperedges $K = \{e_1, e_2, ..., e_k\}$ satisfying:

[5] Because f° does not appear in any of the generated hypergraphs.

- Each e_i, $i \in [k]$, is taken from a cluster of $\bar{C}^*$ or a non-root cluster[6] of C_z^* for some $z \in Z$.
- No two hyperedges in K are taken from the same cluster.

Before delving into the next lemma, the reader should recall that $\{\text{sigpath}(f, T) \mid f \in F\}$ is a clustering of E.

Lemma 8 (Distinct Clusters Lemma 2)**.** If $\{e_1, e_2, ..., e_k\}$ is a super-k-group, then $\{e_1, e_2, ..., e_k\}$ must be a k-group of the clustering $\{\text{sigpath}(f, T) \mid f \in F\}$.

Example 2.10. In Figure 5, $C^*_{\texttt{CEF}} = \{\{\texttt{EFG}, \texttt{CEF}\}, \{\texttt{CEJ}\}\}$, $C^*_{\texttt{BO}} = \{\{\texttt{BO}\}, \{\texttt{BCE}\}\}$, $C^*_{\texttt{BD}} = \{\{\texttt{BD}\}, \{\texttt{BCE}\}\}$, $\bar{C}^* = \{\{\texttt{HI}\}, \{\texttt{EHJ}\}, \{\texttt{HK}\}, \{\texttt{HN}\}, \{\texttt{LM}, \texttt{KL}\}\}$. A super-4-group is {CEF, BO, BD, KL}. Lemma 8 assures us that {CEF, BO, BD, KL} must be a 4-group in the clustering C given in Example 2.4. □

3 An MPC Algorithm

The rest of the paper will apply the theory of CECs to solve acyclic queries in the MPC model. We will describe a variant of Hu's algorithm [8] in this section[7] and present our analysis in the next section. Denote by Q the acyclic query to be answered. Let $G = (V, E)$ be the hypergraph of Q. We assume G to be clean; otherwise, Q can be converted to a clean query having the same result with load $O(m/p)$ [8]. We will also assume that Q has at least two relations; otherwise, the query is trivial and requires no communication.

3.1 Configurations

Let T be a hyperedge tree of G and F be the CEC of G induced by T. The size of F is precisely ρ, the fractional edge covering number of Q (Section 1.2). As explained in Section 2.1, when G is clean,

$$C = \{\text{sigpath}(f, T) \mid f \in F\} \tag{5}$$

is a clustering of E. Let f° be an anchor leaf of T and A° an anchor attribute of f° (Section 2.1); remember that A° appears in all the hyperedges of $\text{sigpath}(f^\circ, T)$. Define

$$L = \text{the } Q\text{-induced load of } C. \tag{6}$$

The reader can review Equation (1) for the definition of "Q-induced load".

For each hyperedge $e \in E$, as before $R(e)$ denotes the relation in Q corresponding to e. Fix a value $x \in \mathbf{dom}$. Given an $e \in \text{sigpath}(f^\circ, T)$, we define the *$A^\circ$-frequency* of x in $R(e)$ as the number of tuples $\boldsymbol{u} \in R(e)$ such that $\boldsymbol{u}(A^\circ) = x$. Further define the *signature-path A°-frequency* of x as the sum of its A°-frequencies in the $R(e)$ of all $e \in \text{sigpath}(f^\circ, T)$. A value $x \in \mathbf{dom}$ is

- *heavy*, if its signature-path A°-frequency is at least L;
- *light*, otherwise.

[6] Namely, e_i cannot be the root of T_z^*.

[7] Our algorithm follows Hu's ideas [8] but differs in certain details. For example, Hu's algorithm takes an arbitrary optimal edge cover of G as the input, while we insist on working with a CEC.

Divide **dom** into disjoint intervals such that the light values in each interval have a total signature-path A°-frequency of $\Theta(L)$. We will refer to those intervals as the *light intervals* of A°. The total number of heavy values and light intervals is at most

$$\sum_{e \in \text{sigpath}(f^\circ, T)} \frac{|R(e)|}{L} = O\left(\max_{e \in \text{sigpath}(f^\circ, T)} \frac{|R(e)|}{L}\right) = O\left(\frac{\max\ (1, Q)\text{-product of } C}{L}\right) = O(p) \quad (7)$$

where the first equality used the fact that sigpath(f°, T) has $O(1)$ edges and the second equality applied the definition of max (k, Q)-product (see Section 1.3).

A *configuration* η is either a heavy value or a light interval. Equation (7) implies that the number of configurations is $O(p)$. For each hypergraph $e \in E$, define a relation $R(e, \eta)$ as follows:

- if η is a heavy value, $R(e, \eta)$ includes all and only the tuples $\boldsymbol{u} \in R(e)$ satisfying $\boldsymbol{u}(A^\circ) = \eta$;
- if η is a light interval, $R(e, \eta)$ includes all and only the tuples $\boldsymbol{u} \in R(e)$ where $\boldsymbol{u}(A^\circ)$ is a light value in η.

Note that $R(e, \eta) = R(e)$ if $A^\circ \notin e$. Let Q_η be the query defined by $\{R(e, \eta) \mid e \in E\}$. Our objective is to compute $Join(Q_\eta)$ for all η in parallel. The final result $Join(Q)$ is simply $\bigcup_\eta Join(Q_\eta)$.

The rest of the section will explain how to solve $Join(Q_\eta)$ for an arbitrary η. We allocate

$$p_\eta \quad = \quad \Theta\left(1 + \max_{k=1}^{|F|} \frac{P_k(Q_\eta, C)}{L^k}\right) \quad (8)$$

machines for this purpose, where $P_k(Q_\eta, C)$ is the max (k, Q_η)-product of C.

3.2 Solving Q_η When η is a Heavy Value

Define the residual hypergraph $G' = (V', E')$ after removing A°, and also functions $map(.)$ and $map^{-1}(.)$ as in Section 2.2.1. We compute $Join(Q_\eta)$ in five steps.

Step 1. Send the tuples of $R(e, \eta)$, for all $e \in E$, to the p_η allocated machines such that each machine receives $\Theta(\frac{1}{p_\eta} \sum_{e \in E} |R(e, \eta)|)$ tuples.

Step 2. For each $e \in E$, convert $R(e, \eta)$ to $R^*(e', \eta)$ where $e' = map(e) = e \setminus \{A^\circ\}$. Specifically, $R^*(e', \eta)$ is a copy of $R(e, \eta)$ but with A° discarded, or formally, $R^*(e', \eta) = \{\boldsymbol{u}[e'] \mid \text{tuple } \boldsymbol{u} \in R(e, \eta)\}$. No communication occurs as each machine simply discards A° from every tuple $\boldsymbol{u} \in R(e, \eta)$ in the local storage.

Step 3. Cleanse G' into $G^* = (V', E^*)$. As explained in Section 2.2.1, this may or may not require calling algorithm cleanse. If called, cleanse identifies in each iteration two hyperedges e_{small} and e_{big} in the current G^* and removes e_{small}. Accordingly, we perform a *semi-join* between $R^*(e_{\text{small}}, \eta)$ and $R^*(e_{\text{big}}, \eta)$, which removes every tuple $\boldsymbol{u}$ from $R^*(e_{\text{big}}, \eta)$ with the property that $\boldsymbol{u}[e_{\text{small}}]$ is absent from $R^*(e_{\text{small}}, \eta)$. $R^*(e_{\text{small}}, \eta)$ is discarded after the semi-join.

Step 4. Let Q^*_η be the query defined by the relation set $\{R^*(e^*, \eta) \mid e^* \in E^*\}$. Compute $Join(Q^*_\eta)$ using p_η machines recursively. Note that the number of participating attributes has decreased by 1 for the recursion.

Step 5. We output $Join(Q_\eta)$ by augmenting each tuple $\boldsymbol{u} \in Join(Q^*_\eta)$ with $\boldsymbol{u}(A^\circ) = \eta$. No communication is needed.

3.3 Solving Q_η When η is a Light Interval

Define Z, $G_z^* = (V_z^*, E_z^*)$ (for each $z \in Z$), C_z^*, $\bar{G}^* = (\bar{V}^*, \bar{E}^*)$, and $\bar{C}^*$ all in the way described in Section 2.2.2. We compute $Join(Q_\eta)$ in four steps.

Step 1. Same as Step 1 of the algorithm in Section 3.2.

Step 2. For each $e \in \text{sigpath}(f^\circ, T)$, broadcast $R(e, \eta)$ to all p_η machines. By definition of light interval, the size of $R(e, \eta)$ is at most L.

Step 3. For each $z \in Z$, define a query $Q_{\eta,z}^* = \{R(e, \eta) \mid e \in E_z^*\}$. Similarly, for $\bar{G}^*$, define a query $\bar{Q}_\eta^* = \{R(e, \eta) \mid e \in \bar{E}^*\}$. Next, we compute the cartesian product of $Join(\bar{Q}_\eta^*)$ and the $Join(Q_{\eta,z}^*)$ of all the $z \in Z$ — namely $\left(\times_{z \in Z} Join(Q_{\eta,z}^*)\right) \times Join(\bar{Q}_\eta^*)$ — using p_η machines. Towards that purpose, define for each $z \in Z$

$$p_{\eta,z} = \Theta\left(1 + \max_{k=1}^{|F_z^*|} \frac{P_k(Q_{\eta,z}^*, C_z^*)}{L^k}\right) \tag{9}$$

where $P_k(Q_{\eta,z}^*, C_z^*)$ is the max $(k, Q_{\eta,z}^*)$-product of the clustering C_z^*. Similarly, define

$$\bar{p}_\eta = \Theta\left(1 + \max_{k=1}^{|\bar{F}^*|} \frac{P_k(\bar{Q}_\eta^*, \bar{C}^*)}{L^k}\right) \tag{10}$$

where $P_k(\bar{Q}_\eta^*, \bar{C}^*)$ is the max $(k, \bar{Q}_\eta^*)$-product of the clustering $\bar{C}^*$. We will prove later that each $Q_{\eta,z}^*$ can be answered with load $O(L)$ using $p_{\eta,z}$ machines, and $\bar{Q}_\eta^*$ can be answered with load $O(L)$ using $\bar{p}_\eta$ machines. Therefore, applying the cartesian product algorithm given in Lemma 6 of [12] (see also Lemma 4 of [13]), we can compute $\left(\times_{z \in Z} Join(Q_{\eta,z}^*)\right) \times Join(\bar{Q}_\eta^*)$ with load $O(L)$ using $\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z}$ machines. As proved later, we can adjust the constants in (9) and (10) to make sure $\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} \leq p_\eta$, where p_η is given in (8).

Step 4. We combine the cartesian product $\left(\times_{z \in Z} Join(Q_{\eta,z}^*)\right) \times Join(\bar{Q}_\eta^*)$ with the tuples broadcast in Step 2 to derive $Join(Q_\eta)$ with no more communication. Specifically, for each tuple $\boldsymbol{u}$ in the cartesian product, the machine where $\boldsymbol{u}$ resides outputs $\{\boldsymbol{u}\} \bowtie \left(\bowtie_{e \in \text{sigpath}(f^\circ, T)} R(e, \eta)\right)$. It is rudimentary to verify that all the tuples of $Join(Q_\eta)$ will be produced this way.

4 Analysis of the Algorithm

This section will establish:

Theorem 9. Consider any join query Q defined in Section 1.1 whose hypergraph is G. The algorithm of Section 3 answers Q with load $O(L)$, where L (given in (6)) is the Q-induced load of the clustering obtained from a canonical edge cover of G.

We will prove the theorem by induction on the number of participating attributes (i.e., $|V|$) and the number of participating relations (i.e., $|Q|$). If $|Q| = 1$, the theorem trivially holds. If $|V| = 1$, Q has only one relation (because Q is clean) and the theorem also holds. Next, assuming that the theorem holds on any query with *either* strictly less participating attributes *or* strictly less participating relations than Q, we will prove the theorem's correctness on Q.

Our analysis will answer three questions. First, why do we have enough machines to handle all configurations in parallel? In particular, we must show that $\sum_\eta p_\eta \leq p$, where p_η is given in (8). Second, why does each step in Section 3.2 and 3.3 entail a load of $O(L)$? Third, why do we have

$\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} \leq p_\eta$ in Step 3 of Section 3.3? Settling these questions will complete the proof of Theorem 9.

All the notations in this section follow those in Section 3.

4.1 Total Number of Machines for All Configurations

It suffices to prove $\sum_\eta p_\eta = O(p)$ because adjusting the hidden constants then ensures $\sum_\eta p_\eta \leq p$. For every $k \in [|F|]$, we will show

$$\frac{1}{L^k} \sum_\eta P_k(Q_\eta, C) \quad = \quad O(p) \tag{11}$$

which will yield

$$\begin{aligned} \sum_\eta p_\eta &= \sum_\eta O\left(1 + \max_{k=1}^{|F|} \frac{P_k(Q_\eta, C)}{L^k}\right) \\ &= \sum_\eta O\left(1 + \sum_{k=1}^{|F|} \frac{P_k(Q_\eta, C)}{L^k}\right) = O(p) + \sum_{k=1}^{|F|} O\left(\sum_\eta \frac{P_k(Q_\eta, C)}{L^k}\right) \\ &= O(p) \end{aligned}$$

where the second equality used $|F| = O(1)$ and the third equality used $\sum_\eta 1 = O(p)$.[8]

Henceforth, fix the value of k. For any η, the hypergraph of Q_η is always G (i.e., the hypergraph of Q). Consider an arbitrary k-group K of the clustering C (given in Equation 5). The Q_η-product of K is $\prod_{e \in K} |R(e, \eta)|$.[9] For any K, we will prove

$$\frac{1}{L^k} \sum_\eta \prod_{e \in K} |R(e, \eta)| \quad = \quad O(p). \tag{12}$$

As C has $O(1)$ k-groups K, the above yields

$$\begin{aligned} \sum_\eta \frac{P_k(Q_\eta, C)}{L^k} &= \sum_\eta \frac{1}{L^k} \max_K \prod_{e \in K} |R(e, \eta)| \\ &= O\left(\sum_\eta \frac{1}{L^k} \sum_K \prod_{e \in K} |R(e, \eta)|\right) = O\left(\sum_K \frac{1}{L^k} \sum_\eta \prod_{e \in K} |R(e, \eta)|\right) \\ &= \sum_K O(p) = O(p) \end{aligned}$$

as claimed in (11).

Let us first consider the case where $K \cap \text{sigpath}(f^\circ, T) \neq \emptyset$, namely, K has a hyperedge e_0 picked from the cluster $\text{sigpath}(f^\circ, T)$. We have:

$$\sum_\eta \prod_{e \in K} |R(e, \eta)| \quad = \quad \sum_\eta \left(|R(e_0, \eta)| \cdot \prod_{e \in K \setminus \{e_0\}} |R(e, \eta)|\right) \tag{13}$$

[8] $\sum_\eta 1$ is the number of configurations which is $O(p)$ as shown in (7).

[9] For the definition of "a k-group's Q-product", review Section 1.3.

For each $e \in K \setminus \{e_0\}$, obviously $|R(e,\eta)| \leq |R(e)|$. Regarding e_0, because A° must be an attribute of e_0, the relations $R(e_0,\eta)$ of all the configurations η form a *partition* of $R(e_0)$.[10] Hence:

$$\begin{aligned}(13) \quad &\leq \Big(\prod_{e\in K\setminus\{e_0\}} |R(e)|\Big)\Big(\sum_{\eta} |R(e_0,\eta)|\Big) = \Big(\prod_{e\in K\setminus\{e_0\}} |R(e)|\Big)\cdot|R(e_0)| = \prod_{e\in K}|R(e)| \\ &\leq \text{ max } (k,Q)\text{-product of } C.\end{aligned}$$

Therefore, the left hand side of (12) is bounded by $\frac{\text{max } (k,Q)\text{-product of } C}{L^k}$, which is at most p (by definition of L).

Next, we consider $K \cap \text{sigpath}(f^\circ, T) = \emptyset$. In this case, we must have $k = |K| \leq |F| - 1$, because the hyperedges in K need to come from distinct clusters of C, and C has $|F|$ clusters (one of them is $\text{sigpath}(f^\circ, T)$, which now must be excluded). Applying the trivial fact $|R(e,\eta)| \leq |R(e)|$ (for any e) and the fact that $\sum_\eta 1$ is bounded by (7), we have

$$\begin{aligned}\frac{1}{L^k}\sum_{\eta}\prod_{e\in K}|R(e,\eta)| &\leq \frac{1}{L^k}\sum_{\eta}\prod_{e\in K}|R(e)| = O\left(\frac{1}{L^k}\prod_{e\in K}|R(e)|\cdot\max_{e\in\text{sigpath}(f^\circ,T)}\frac{|R(e)|}{L}\right) \\ &= O\left(\frac{\text{max } (k+1,Q)\text{-product of } C}{L^{k+1}}\right)\end{aligned}$$

which is at most p. This completes the proof of $\sum_\eta p_\eta = O(p)$.

4.2 Heavy Q_η

This subsection will prove that the algorithm in Section 3.2 has load $O(L)$. Step 2 and 5 demand no communication. The loads of Step 1 and 3 can all be bounded[11] by $O(\frac{1}{p_\eta}\sum_{e\in E}|R(e,\eta)|) = O(\frac{1}{p_\eta}\max_{e\in E}|R(e,\eta)|) = O(P_1(Q_\eta, C)/p_\eta) = O(L)$.

To analyze Step 4, let T^* be the hyperedge tree of G^* (produced by cleansing) and F^* be the CEC of G^*. By definition, the Q^*_η-induced load of the clustering $C^* = \{\text{sigpath}(f^*, T^*) \mid f^* \in F^*\}$ is

$$L^*_\eta = \max_{k=1}^{|F^*|}\left(\frac{P_k(Q^*_\eta, C^*)}{p_\eta}\right)^{1/k} \tag{14}$$

where $P_k(Q^*_\eta, C^*)$ is the max (k, Q^*_η)-product of C^*. By our inductive assumption (that Theorem 9 holds on Q^*_η), Step 4 incurs load $O(L^*_\eta)$. We will prove $P_k(Q^*_\eta, C^*) \leq P_k(Q_\eta, C)$ for every k which, together with (8) and (14), will tell us $L^*_\eta = O(L)$.

Before proceeding, the reader should recall that, for any hyperedge e^* of G^*, $map^{-1}(e^*)$ gives a hyperedge in G. We must have $|R^*(e^*,\eta)| \leq |R(map^{-1}(e^*),\eta)|$. To see why, note that this is true when $|R^*(e^*,\eta)|$ is created in Step 2, whereas $R^*(e^*,\eta)$ can only shrink in Steps 3-5.

To prove $P_k(Q^*_\eta, C^*) \leq P_k(Q_\eta, C)$, consider any k-group K^* of C^*. By Lemma 6, $K = \{map^{-1}(e^*) \mid e^* \in K^*\}$ must be a k-group of C. Since $|R^*(e^*,\eta)| \leq |R(map^{-1}(e^*),\eta)|$ for any $e^* \in K^*$, we have $\prod_{e^*\in K^*}|R^*(e^*,\eta)| \leq \prod_{e\in K}|R(e,\eta)| \leq P_k(Q_\eta, C)$. Therefore:

$$P_k(Q^*_\eta, C^*) = \max_{K^*}\prod_{e^*\in K^*}|R^*(e^*,\eta)| \leq P_k(Q_\eta, C).$$

[10] The $R(e_0,\eta)$ of all the η are mutually disjoint and their union equals $R(e_0)$.

[11] Step 3 performs $O(1)$ semi joins, each of which can be performed by sorting. For sorting in the MPC model, see Section 2.2.1 of [10]. The stated bound for Step 1 and 3 requires the assumption $p \leq m^{1-\epsilon}$.

4.3 Light Q_η

This subsection will concentrate on the algorithm of Section 3.3.

Load. Step 1 incurs load $O(L)$ (same analysis as in Section 3.2). Step 2 also requires a load of $O(L)$ because every broadcast relation has a size of at most L. Step 4 needs no communication.

To analyze Step 3, let us first consider $\bar{Q}^*_\eta$. The $\bar{Q}^*_\eta$-induced load of the clustering $\bar{C}^*$ is

$$\bar{L}^*_\eta = \max_{k=1}^{|\bar{C}^*|} \left(\frac{P_k(\bar{Q}^*_\eta, \bar{C}^*)}{\bar{p}_\eta} \right)^{1/k}$$

where $P_k(\bar{Q}^*_\eta, \bar{C}^*)$ as the max $(k, \bar{Q}^*_\eta)$-product of $\bar{C}^*$. By our inductive assumption (that Theorem 9 holds on $\bar{Q}^*_\eta$), answering $\bar{Q}^*_\eta$ with $\bar{p}_\eta$ machines requires load $O(\bar{L}^*_\eta)$, which is $O(L)$ given the $\bar{p}_\eta$ in (10). A similar argument shows that answering each $Q^*_{\eta,z}$ with $p_{\eta,z}$ machines — with $p_{\eta,z}$ given in (9) — incurs a load of $O(L)$. Thus, the cartesian product at Step 3 can be computed with load $O(L)$.

Number of machines in Step 3. Next, we will prove that $\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} \leq p_\eta$ always holds in Step 3. It suffices to show $\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} = O(p_\eta)$ which, as we will see, relies on Lemma 8 and the fact that $|R(e, \eta)| \leq L$ for every node $e \in \text{sigpath}(f^\circ, T)$.

Consider an arbitrary $z \in Z$. The root of T^*_z — denoted as e_{root} — must belong to $\text{sigpath}(f^\circ, T)$. Recall that a k-group K of C^*_z takes a hyperedge from a distinct cluster in C^*_z. Call K a *non-root k-group* if $e_{root} \notin K$, or a *root k-group*, otherwise. Define

$$\begin{aligned} P_k(Q^*_{\eta,z}, C^*_z) &= \max\ (k, Q^*_{\eta,z})\text{-product of } C^*_z \\ P^{non}_k(Q^*_{\eta,z}, C^*_z) &= \max\ (k, Q^*_{\eta,z})\text{-product of all the } \textit{non-root } k\text{-groups of } C^*_z. \end{aligned}$$

As a special case, define $P^{non}_0(Q^*_{\eta,z}, C^*_z) = 1$. For any k, we observe

$$P_k(Q^*_{\eta,z}, C^*_z) \leq \max\{P^{non}_k(Q^*_{\eta,z}, C^*_z), L \cdot P^{non}_{k-1}(Q^*_{\eta,z}, C^*_z)\}. \tag{15}$$

To prove the inequality, fix K to the k-group with the largest $Q^*_{\eta,z}$-product ($= P_k(Q^*_{\eta,z}, C^*_z)$). If K is a non-root k-group, (15) obviously holds. Consider, instead, that K is a root k-group. Since $e_{root} \in \text{sigpath}(f^\circ, T)$, we know $|R(e_{root}, \eta)| \leq L$ and hence $\prod_{e \in K} |R(e, \eta)| \leq L \cdot \prod_{e \in K \setminus \{e_{root}\}} |R(e, \eta)|$. As $K \setminus \{e_{root}\}$ is a non-root $(k-1)$-group, $P_k(Q^*_{\eta,z}, C^*_z) \leq L \cdot P^{non}_{k-1}(Q^*_{\eta,z}, C^*_z)$ holds.

Equipped with (15), we can now derive from (9):

$$\begin{aligned} p_{\eta,z} &= O\left(1 + \max_{k=1}^{|F^*_z|} \frac{\max\{P^{non}_k(Q^*_{\eta,z}, C^*_z), L \cdot P^{non}_{k-1}(Q^*_{\eta,z}, C^*_z)\}}{L^k}\right) \\ &= O\left(1 + \max_{k=1}^{|F^*_z|-1} \frac{P^{non}_k(Q^*_{\eta,z}, C^*_z)}{L^k}\right) \end{aligned} \tag{16}$$

where the second equality used the fact that, when $k = |F^*_z|$, a k-group must be a root k-group.

We are now ready to prove $\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} = O(p_\eta)$. For each $z \in Z$, define integer k_z and a set K_z of hyperedges as follows:

- If (16) $= \Theta(P^{non}_k(Q^*_{\eta,z}, C^*_z)/L^k)$ for some $k \in [1, |F^*_z| - 1]$, set $k_z = k$ and K_z to the non-root k-group whose $Q^*_{\eta,z}$-product equals $P^{non}_k(Q^*_{\eta,z}, C^*_z)$.

- Otherwise (we must have $p_{\eta,z} = \Theta(1)$), set $k_z = 0$ and $K_z = \emptyset$; furthermore, define the $Q^*_{\eta,z}$-product of K_z to be 1.

Similarly, regarding $\bar{p}_\eta$ in (10), define integer $\bar{k}$ and a set $\bar{K}$ of hyperedges as follows:

- If (10) $= \Theta(P_k(\bar{Q}^*_\eta, \bar{C}^*)/L^k)$ for some $k \in [1, |\bar{F}^*|]$, set $\bar{k} = k$ and $\bar{K}$ to the k-group of the clustering $\bar{C}^*$ whose $\bar{Q}^*_\eta$-product equals $P_k(\bar{Q}^*_\eta, \bar{C}^*)$.
- Otherwise, set $\bar{k} = 0$ and $\bar{K} = \emptyset$; furthermore, define the $\bar{Q}^*_\eta$-product of $\bar{K}$ to be 1.

Define $K_{super} = \bar{K} \cup \left(\bigcup_{z \in Z} K_z\right)$. If $K_{super} = \emptyset$, then $p_{\eta,z} = \Theta(1)$ for all $z \in Z$ and $\bar{p}_\eta = \Theta(1)$, which leads to

$$\bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} = O(1) = O(p_\eta).$$

If $K_{super} \neq \emptyset$, K_{super} is a super-$|K_{super}|$-group[12]. By Lemma 8, K_{super} is a $|K_{super}|$-group of T. We thus have:

$$\begin{aligned} \bar{p}_\eta \cdot \prod_{z \in Z} p_{\eta,z} &= \frac{\bar{Q}^*_\eta\text{-product of } \bar{K}}{L^{\bar{k}}} \prod_{z \in Z} \frac{Q^*_{\eta,z}\text{-product of } K_z}{L^{k_z}} \\ &= \frac{\prod_{e \in K_{super}} |R(e)|}{L^{|K_{super}|}} \leq \frac{\max\ (|K_{super}|, Q_\eta)\text{-product of } C}{L^{|K_{super}|}} = O(p_\eta). \end{aligned}$$

This completes the whole proof of Theorem 9.

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Appendix

A Proof of Lemma 1

We first show that F is an edge cover of G. Each attribute $X \in V$ is a disappearing attribute of some hyperedge $e \in E$. When e is processed at Line 4 of edge-cover, either X is already covered or e itself will be added to F_{tmp} (which will then cover X).

Next, we argue that F is an *optimal* edge cover (i.e., having the smallest size). Let F' be an arbitrary optimal edge cover of G. We will establish a one-one mapping between F and F', which implies the optimality of F.

Fix an arbitrary hyperedge $e \in F$. If e also belongs to F', we map e to its copy in F'. Consider the opposite case where $e \notin F'$. The fact $e \in F$ indicates that when edge-cover processes e, e must contain a disappearing attribute X that has not been covered by F_{tmp}. Let $e' \in F'$ be an arbitrary hyperedge containing X; we map e to e'. As explained in Section 2.1, e' must be a proper descendant of e in T.

We argue that no two e and $\hat{e}$ in F can be mapped to the same hyperedge $e' \in F$. If this happens, e' is a descendant of both e and $\hat{e}$. Assume, without loss of generality, that e is a proper descendant of $\hat{e}$. Since $\hat{e}$ is mapped to e', there is an attribute Y such that

- Y is a disappearing attribute in $\hat{e}$ not covered by F_{tmp} when edge-cover adds $\hat{e}$ to F_{tmp};
- $Y \in e'$.

Because e is on the path from $\hat{e}$ to e' in T, connectedness of acyclicity guarantees $Y \in e$. On the other hand, $e \in F$ and e is processed before $\hat{e}$ (reverse topological order). Thus, when $\hat{e}$ is processed, $e \in F_{\text{tmp}}$ and hence Y must be covered by F_{tmp}, giving a contradiction.

We now proceed to show that F does not depend on the reverse topological order at Line 2. Recall that, when processing a node e, edge-cover adds it to F_{tmp} if and only if F_{tmp} does not cover a disappearing attribute X of e. All the nodes containing X must appear in the subtree of T rooted at e and thus must be processed before e. Hence, whether $e \in F$ is determined by which of those nodes are selected into F_{tmp}. The observation gives rise to an inductive argument. First, if e is a leaf, e enters F_{tmp} if and only if it has a disappearing attribute (which must be exclusive), independent of the reverse topological order used. For a non-leaf node e, inductively, once we have decided whether $e' \in F_{\text{tmp}}$ for every proper descendent e' of e, whether $e \in F_{\text{tmp}}$ has also been decided. We thus conclude that the reverse topological order has no influence on the output.

It remains to show that when G is clean, F must include all the raw leaf nodes e of T. If e is not the root of T, it must have an attribute X absent from the parent node of e (otherwise, e is subsumed by its parent and G is not clean). Similarly, if e is the root of T, it must have an attribute X absent from its child (there is only one child because e is a raw leaf). In both cases, the attribute X is exclusive at e and will force edge-cover to add e to F_{tmp}.

B Proof of Lemma 2

Identify an arbitrary non-leaf node $\hat{f} \in F$ such that no other non-leaf node in F is lower than $\hat{f}$. The existence of $\hat{f}$ is guaranteed because F includes the root of T. Consider any child node e of $\hat{f}$ in T. Since G is clean, e must have an attribute A° that does not appear in $\hat{f}$. Let f° be any node

in F that contains A°. By the connectedness requirement of acyclicity, f° must be in the subtree of T rooted at e and, therefore, must be a leaf.

We argue that f° is an anchor leaf. The signature path of f° includes all the nodes on the path from e to f°. Because $A^\circ \in e$ and $A^\circ \in f^\circ$, A° must appear in all the nodes on the path (connectedness requirement) and is thus an anchor attribute of f°.

C Proof of Lemma 3

C.1 $map(f^\circ)$ Subsumed in G'

Let $\hat{e}$ be the parent of f° in T. If $map(f^\circ) = f^\circ \setminus \{A^\circ\}$ is subsumed in G', then $map(f^\circ)$ must be a subset of $map(\hat{e})$, which indicates $A^\circ \notin \hat{e}$ (otherwise, $f^\circ \subseteq \hat{e}$ and G is not clean). Because A° needs to appear in all the nodes of sigpath(f°, T), $A^\circ \notin \hat{e}$ indicates that sigpath(f°, T) has only a single node f°. It thus follows that $\hat{e} \in F$ and A° is an exclusive attribute in f°. Hence, the removal of A° does not affect any hyperedge except f°.

Next, we show that $F' = F \setminus \{f^\circ\}$ is the CEC of G' induced by T'. It suffices to prove that F' is the output of edge-cover(T') on some reverse topological order of T'. For this purpose, consider σ_0 as an arbitrary reverse topological order of T where $\hat{e}$ succeeds f°. Let σ_1 be the sequence obtained by removing f° from σ_0; σ_1 must be a reverse topological order of T'. Let e_{before} be the node preceding f° in σ_0 (and hence preceding $\hat{e}$ in σ_1); define e_{before} to be a dummy node if f° is the first in σ_0.

Let us compare the execution of edge-cover(T) on σ_0 to that of edge-cover(T') on σ_1. The two executions are identical till the moment when e_{before} has been processed. By the fact that edge-cover(T) adds $\hat{e}$ to F_{tmp} (we have proved earlier $\hat{e} \in F$), $\hat{e}$ has a disappearing attribute not covered by F_{tmp} when $\hat{e}$ is processed. Hence, when $\hat{e}$ is processed by edge-cover(T'), it must also have a disappearing attribute not covered by F_{tmp} and thus is added to F_{tmp}. The rest execution of edge-cover(T) is the same as that of edge-cover(T') because every non-exclusive attribute of f° is in $\hat{e}$. Therefore, the output of edge-cover(T') is the same as that of edge-cover(T), except that the former does not include f°.

C.2 $map(f^\circ)$ Not Subsumed in G'

Let $\sigma_0 = (e_1, e_2, ..., e_{|E|})$ be an arbitrary reverse topological order of T. Define $e'_i = map(e_i) = e_i \setminus \{A^\circ\}$ for $i \in [|E|]$. The sequence $\sigma_1 = (e'_1, e'_2, ..., e'_{|E|})$ is a reverse topological order of T'. We will compare the execution of edge-cover(T) on σ_0 to that of edge-cover(T') on σ_1. Define $F_0(e_i)$ (resp., $F_1(e'_i)$) as the content of F_{tmp} after edge-cover(T) (resp., edge-cover(T')) has processed e_i (resp., e'_i).

Claim 1: For any leaf e of T, *edge-cover*(T') must add $e' = map(e)$ to F_{tmp}.

Let us prove the claim. Because e is a leaf of T and G is clean, e must have an exclusive attribute X. If edge-cover does not add e' to F_{tmp}, e' has no exclusive attributes in T'. This implies $X = A^\circ$, which further implies $f^\circ = e$ (otherwise, A° appears in two distinct nodes and cannot be exclusive). However, in that case, e' must contain an exclusive attribute in T' ($e' = map(f^\circ)$ is not subsumed in G'). We thus have reached a contradiction.

To establish Lemma 3, it suffices to prove:

Claim 2: For each i, $e_i \in F_0(e_i)$ if and only if $e'_i \in F_1(e'_i)$.

We prove the claim by induction on i. Because e_1 is a leaf of T, Lemma 1 and Claim 1 guarantee $e_1 \in F_0(e_1)$ and $e'_1 \in F_1(e'_1)$, respectively. Thus, Claim 2 holds for $i = 1$.

Next, we prove the correctness on $i > 1$, assuming that it holds on e_{i-1} and e'_{i-1}. The inductive assumption implies that $F_0(e_{i-1})$ covers an attribute $X \neq A^\circ$ if and only if $F_1(e'_{i-1})$ covers X. If $e_i \notin F_0(e_i)$, every disappearing attribute of e_i must be covered by $F_0(e_{i-1})$. Hence, $F_1(e'_{i-1})$ must cover all the disappearing attributes of e'_i and thus $e'_i \notin F_1(e'_i)$.

The rest of the proof assumes $e_i \in F_0(e_i)$, i.e., e_i has a disappearing attribute X not covered by $F_0(e_{i-1})$. If $X \neq A^\circ$, X is a disappearing attribute in e'_i not covered by $F_1(e'_{i-1})$ and thus $e'_i \in F_1(e'_i)$. It remains to discuss the scenario $X = A^\circ$. As A° is disappearing at e_i, A° cannot exist in the parent of e_i. On the other hand, because $A^\circ \in f^\circ$, acyclicity's connectedness requirement forces f° to be a descendant of e_i. We can safely conclude that $f^\circ = e_i$; otherwise, the leaf f° is processed before e_i and must exist in $F_0(e_{i-1})$ (Lemma 1), contradicting the fact that A° is not covered by $F_0(e_{i-1})$. Then, $e'_i \in F_1(e'_i)$ follows from Claim 1.

D Proof of Lemma 4

Define $e = map^{-1}(e')$. Because e' is subsumed, we know that e must contain A° (otherwise, e is subsumed and G is not clean). In other words, $e = e' \cup \{A^\circ\}$. Furthermore, if $e = f^\circ$, then $map(f^\circ) = map(e) = e'$ is subsumed in G', in which case we must have $e' \notin F'$ (by the way we define F' in (2)). Next, we assume $e \neq f^\circ$. To complete the proof, it suffices to show that $e \notin F$, where F is the CEC of G induced by T.

Let $\hat{f}$ be the lowest proper ancestor of f° in F (here, "ancestor" is defined with respect to T). By definition of A°, $A^\circ \notin \hat{f}$. Because $A^\circ \in f^\circ$ and $A^\circ \in e$, e must be a proper descendant of $\hat{f}$ in T. Assume, for contradiction purposes, that $e \in F$. As $\hat{f}$ (by definition of f°) cannot have any non-leaf proper descendant in F, e must be a leaf of T.

Because A° appears in two distinct nodes e and f°, acyclicity's connected requirement demands that A° should also exist in the parent $\hat{e}$ of e. Because G is clean, we know that e must have at least one attribute X that does not appear in $\hat{e}$ and thus must be exclusive. It follows that $X \neq A^\circ$. However, in that case $e' = e \setminus \{A^\circ\}$ contains X and thus cannot be subsumed in G' (X remains exclusive in G'), giving a contradiction.

E Proof of Lemma 5

We discuss only the scenario where $map(f^\circ)$ is not subsumed in G' (the opposite case is easy and omitted). Our proof will establish a stronger claim:

Claim: $F^* = F'$ is the CEC of G^* induced by T^* every time Line 2 of cleanse is executed.

$G^* = G'$ and $T^* = T'$ at Line 1. $F^* = F'$ is the CEC of G^* induced by T^* at this moment (Lemma 3). Hence, the claim holds on the first execution of Line 2.

Inductively, assuming that the claim holds currently, we will show that it still does after cleanse deletes the next e_{small} from G^*. Let G^*_0 and T^*_0 (resp., G^*_1 and T^*_1) be the G^* and T^* before (resp., after) the deletion of e_{small}, respectively. The fact e_{small} being subsumed in G^* suggests e_{small} being subsumed in G'. By Lemma 4, $e_{\text{small}} \notin F' = F^*$.

Case 1: e_{big} parents e_{small}. Let σ_0 be a reverse topological order of T_0^* where e_{big} succeeds e_{small}. As F^* is the CEC of G_0^* induced by T_0^*, edge-cover(T_0^*) produces F^* if executed on σ_0 (Lemma 1).

Let σ_1 be a copy of σ_0 but with e_{small} removed; σ_1 is a reverse topological order of T_1^*. Every node in T_1^* retains the same disappearing attributes as in T_0^* (see Figure 3a). For every node $e \neq e_{\text{small}}$, running edge-cover(T_1^*) on σ_1 has the same effect as running edge-cover(T_0^*) on σ_0. Therefore, edge-cover(T_1^*) also outputs F^*.

Case 2: e_{small} parents e_{big}. Let σ_0 be a reverse topological order of T_0^* where e_{small} succeeds e_{big}. Let σ_1 be a copy of σ_0 but with e_{small} removed; σ_1 is a reverse topological order of T_1^*. We will argue that running edge-cover(T_1^*) on σ_1 returns F^*.

The reader should note several preliminary facts about disappearing attributes. If an attribute has e_{small} as the summit in T_0^*, the attribute's summit in T_1^* becomes e_{big} (see Figure 3b). If an attribute has $e \neq e_{\text{small}}$ as the summit in T_0^*, its summit in T_1^* is still e. Hence, every node in T_1^* except e_{big} retains the same disappearing attributes as in T_0^*, whereas the disappearing attributes of e_{big} in T_1^* contain those of e_{big} and e_{small} in T_0^*.

For each node e in σ_0 (resp. σ_1), denote by $F_0(e)$ (resp. $F_1(e)$) the content of F_{tmp} after edge-cover(T_0^*) (resp. edge-cover(T_1^*)) has processed e. Let e_{before} be the node before e_{big} in σ_0.[13] It is easy to see that edge-cover(T_0^*) and edge-cover(T_1^*) behave the same way until finishing with e_{before}, which gives $F_0(e_{\text{before}}) = F_1(e_{\text{before}})$. It must hold that $e_{\text{small}} \notin F_0(e_{\text{small}})$.[14] Two possibilities apply to e_{big}:

1. $e_{\text{big}} \in F_0(e_{\text{big}})$. Hence, e_{big} has a disappearing attribute in T_0^* not covered by $F_0(e_{\text{before}})$. This means that e_{big} also has a disappearing attribute in T_1^* not covered by $F_1(e_{\text{before}}) = F_0(e_{\text{before}})$. It follows that $e_{\text{big}} \in F_1(e_{\text{big}})$, meaning $F_1(e_{\text{big}}) = F_0(e_{\text{big}}) = F_0(e_{\text{small}})$.

2. $e_{\text{big}} \notin F_0(e_{\text{big}})$. All the disappearing attributes of e_{big} and e_{small} in T_0^* are covered by $F_0(e_{\text{before}})$. Hence, the disappearing attributes of e_{big} in T_1^* are covered by $F_1(e_{\text{before}}) = F_0(e_{\text{before}})$. Therefore, $e_{\text{big}} \notin F_1(e_{\text{big}})$, meaning $F_0(e_{\text{small}}) = F_0(e_{\text{before}}) = F_1(e_{\text{before}}) = F_1(e_{\text{big}})$.

We now conclude that $F_1(e_{\text{big}}) = F_0(e_{\text{small}})$ always holds. Every remaining node in σ_0 and σ_1 has the same disappearing attributes in T_0^* and T_1^*. The rest execution of edge-cover(T_0^*) is identical to that of edge-cover(T_1^*).

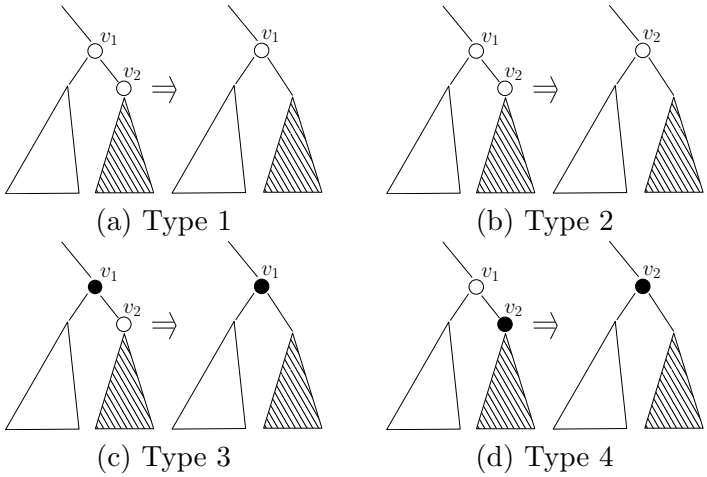
F Proof of Lemma 6

We will discuss only the scenario where $map(f^\circ)$ is not subsumed (the opposite scenario is easy and omitted).

Departing from acyclic queries, let us consider a more general problem on a rooted tree $\mathcal{T}$ where (i) every node is colored black or white, and (ii) the root and all the leaves are black. Denote by B the set of black nodes. Each black node $b \in B$ is associated with a *signature path*:

- If b is the root of $\mathcal{T}$, its signature path contains just b itself.
- Otherwise, let $\hat{b}$ be the lowest ancestor of b among all the nodes in B; the signature path of b is the set of nodes on the path from $\hat{b}$ to b, except $\hat{b}$.

[13] In the special case where e_{big} is the first in σ_0, define e_{before} as a dummy node with $F_0(e_{\text{before}}) = F_1(e_{\text{before}}) = \emptyset$.
[14] Otherwise, $e_{\text{small}} \in F^*$, contradicting Lemma 4.



(a) Type 1 (b) Type 2

(c) Type 3 (d) Type 4

Figure 6: Four types of contraction

We define four types of contractions:

- Type 1: We are given two white nodes v_1 and v_2 such that v_1 parents v_2. The contraction removes v_2 from $\mathcal{T}$ and makes v_1 the new parent for all the child nodes of v_2. See Figure 6a.
- Type 2: We are given two white nodes v_1 and v_2 such that v_1 parents v_2. The contraction removes v_1 from $\mathcal{T}$, makes v_2 the new parent for all the child nodes of v_1, and makes v_2 a child of the original parent of v_1. See Figure 6b.
- Type 3: Same as Type 1, except that v_1 is black and v_2 is white. See Figure 6c.
- Type 4: Same as Type 2, except that v_1 is white and v_2 is black. See Figure 6d.

The facts below are evident:

- The number of black nodes remains the same after a contraction.
- After a contraction, each signature path either remains the same or shrinks.

We now draw correspondence between a contraction and a hyperedge deletion in cleanse. $\mathcal{T}$ corresponds to the current hyperedge tree T^* in cleanse. The set B of black nodes equals $F^* = F'$ for the entire execution of cleanse. The set $\{v_1, v_2\}$ corresponds to $\{e_{\text{small}}, e_{\text{big}}\}$. As shown in Lemma 4, e_{small} cannot exist in F^* and thus cannot correspond to a black node. If we denote by C (resp., C^*) the set of signature paths at the beginning (resp., end) of cleanse, each signature path in C^* is obtained by continuously shrinking a distinct signature path in C. This implies Lemma 6, noticing that $C = \{\text{sigpath}(f, T) \mid f \in F\}$ and $C^* = \{\text{sigpath}(f^*, T^*) \mid f^* \in F^*\}$.

G Proof of Lemma 7

We will first prove that, for any $z \in Z$, F_z^* is the CEC of G_z^* induced by T_z^*. Let $\hat{z}$ be the parent of z. Recall that F is the CEC of G induced by T. Consider a reverse topological order σ_z of T satisfying the following condition: a prefix of σ_z is a permutation of the nodes in the subtree of T rooted at z. In other words, in σ_z, every node in the aforementioned subtree must rank before every node outside the subtree. Define σ_z^* to be the sequence obtained by deleting from σ_z all the

nodes e such that $e \neq \hat{z}$ and e is outside the subtree of T rooted at z. It is clear that σ_z^* is a reverse topological order of T_z^*.

Let us compare the execution of $\mathsf{edge\text{-}cover}(T)$ on σ to that of $\mathsf{edge\text{-}cover}(T_z^*)$ on σ_z^*. They are exactly the same until z has been processed. Hence, every node in the F_{tmp} of $\mathsf{edge\text{-}cover}(T)$ at this moment must have been added to F_{tmp} by $\mathsf{edge\text{-}cover}(T_z^*)$. This means that all the nodes in F_z^*, except $\hat{z}$, must appear in the final F_{tmp} output by $\mathsf{edge\text{-}cover}(T_z^*)$. Finally, the final F_{tmp} must also contain $\hat{z}$ as well due to Lemma 1 (notice that $\hat{z}$ is a raw leaf of T_z^*). This shows that F_z^* is the CEC of G_z^* induced by T_z^*.

Next, we prove that $\bar{F}^*$ is the CEC of $\bar{G}^*$ induced by $\bar{T}^*$. Let $\bar{e}$ be the highest node in $\text{sigpath}(f^\circ, T)$. Consider a reverse topological order $\bar{\sigma}$ of T satisfying the following condition: a prefix of $\bar{\sigma}$ is a permutation of the nodes in the subtree of T rooted at $\bar{e}$. Define $\bar{\sigma}^*$ to be the sequence obtained by deleting that prefix from $\bar{\sigma}$. It is clear that $\bar{\sigma}^*$ is a reverse topological order of $\bar{T}^*$. Define $\hat{\bar{e}}$ to be the parent of $\bar{e}$ in T. Note that $\hat{\bar{e}}$ must belong to F due to the definitions of $\bar{e}$ and $\text{sigpath}(f^\circ, T)$

We will compare the execution of $\mathsf{edge\text{-}cover}(T)$ on σ to that of $\mathsf{edge\text{-}cover}(\bar{T}^*)$ on $\bar{\sigma}^*$. For each e in σ, define $F_0(e)$ as the content of F_{tmp} after $\mathsf{edge\text{-}cover}(T)$ has finished processing e. Similarly, for each e in $\bar{\sigma}^*$, define $F_1(e)$ as the content of F_{tmp} after $\mathsf{edge\text{-}cover}(\bar{T}^*)$ has finished processing e. Divide σ into three segments: (i) σ_1, which includes the prefix of σ ending at (and including) $\bar{e}$, (ii) σ_2, which starts right after σ_1 and ends at (and includes) $\hat{\bar{e}}$, and (iii) σ_3, which is the rest of σ. Note that $\bar{\sigma}^*$ is the concatenation of σ_2 and σ_3.

Claim 1: For any e in σ_2, $e \in F_0(e)$ if and only if $e \in F_1(e)$.

We prove the claim by induction. As the base case, consider e as the first element in σ_2. In $\bar{T}^*$, e must be a leaf and, by Lemma 1, must be in $F_1(e)$. In T, e is either a leaf or $\hat{\bar{e}}$. In the former case, Lemma 1 assures us $e \in F_0(e)$. In the latter case, e is also in $F_0(e)$ because $\hat{\bar{e}} \in F$.

Next, we prove the claim on every other node e in σ_2, assuming the claim's correctness on the node e_{before} preceding e in σ_2. This inductive assumption implies $F_1(e_{\text{before}}) \subseteq F_0(e_{\text{before}})$. If $e \in F_0(e)$, then e has a disappearing attribute X not covered by $F_0(e_{\text{before}})$. As $F_1(e_{\text{before}}) \subseteq F_0(e_{\text{before}})$, $F_1(e_{\text{before}})$ does not cover X, either. Hence, $\mathsf{edge\text{-}cover}(\bar{T}^*)$ adds e to F_{tmp}, namely, $e \in F_1(e)$.

Let us now focus on the case where $e \in F_1(e)$. If $e = \hat{\bar{e}}$, the fact $\hat{\bar{e}} \in F$ indicates $e \in F_0(e)$. Next, we consider $e \neq \hat{\bar{e}}$, meaning that e is a proper descendant of $\hat{\bar{e}}$. The fact $e \in F_1(e)$ suggests that e has a disappearing attribute X not covered by $F_1(e_{\text{before}})$. If $e \notin F_0(e)$, $F_0(e_{\text{before}})$ must have a node e' containing X. Node e' must come from σ_1 (the inductive assumption prohibits e' from appearing in σ_2) and hence must be a descendant of $\bar{e}$. By acyclicity's connectedness requirement, X appearing in both e and e' means that X must belong to $\hat{\bar{e}}$. But this contradicts X disappearing at e. We thus conclude that $e \in F_0(e)$.

Claim 2: For any e in σ_3, $e \in F_0(e)$ if and only if $e \in F_1(e)$.

Claim 1 assures us that $F_1(\hat{\bar{e}}) \subseteq F_0(\hat{\bar{e}})$. Note also that $\hat{\bar{e}}$ belongs to $F_0(\hat{\bar{e}})$ (as explained before, $\hat{\bar{e}} \in F$) and hence also to $F_1(\hat{\bar{e}})$ (Claim 1). Any node $e' \in F_0(\hat{\bar{e}}) \setminus F_1(\hat{\bar{e}})$ must appear in the subtree rooted at $\hat{\bar{e}}$ in T, whereas any node e in σ_3 must be outside that subtree. By acyclicity's connectedness requirement, if e' contains an attribute X in e, then $X \in \hat{\bar{e}}$ for sure. This means that $F_1(\hat{\bar{e}})$ covers a disappearing attribute of e if and only if $F_0(\hat{\bar{e}})$ does so. Therefore, $\mathsf{edge\text{-}cover}(\bar{T}^*)$ processes each node of σ_3 in the same way as $\mathsf{edge\text{-}cover}(T)$. This proves the correctness of Claim 2.

By putting Claim 1 and 2 together, we conclude that edge-cover($\bar{T}^*$) returns all and only the attributes in $\sigma_2 \cup \sigma_3$ output by edge-cover(T). Therefore, the output of edge-cover($\bar{T}^*$) is $F \cap \bar{E}^* = \bar{F}^*$.

H Proof of Lemma 8

For any $f^* \in F_z^*$ and any $z \in Z$ that is not the root of T_z^*, it holds that sigpath$(f^*, T_z^*) \subseteq$ sigpath(f^*, T). Similarly, for any $f^* \in \bar{F}^*$, it holds that sigpath$(f^*, \bar{T}^*) \subseteq$ sigpath(f^*, T). To prove the lemma, it suffices to show that, given a super-k-group $K = \{e_1, ..., e_k\}$, we can always assign each e_i, $i \in [k]$, to a distinct cluster in $\{\text{sigpath}(f, T) \mid f \in F\}$. This is easy: if e_i is picked from sigpath(f^*, T_z^*) for some $z \in F$ and $f^* \in F_z^*$, assign e_i to sigpath(f^*, T); if e_i is picked from sigpath$(f^*, \bar{T}^*)$ for some $f^* \in \bar{F}^*$, assign e_i to sigpath(f^*, T).