9.3 Convergence

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- We then apply this sufficient condition to prove the convergence of the BA algorithm for computing C.
- The convergence of the BA algorithm for computing $R(D_s) - sD_s$ can be proved likewise.

9.3.1 A Sufficient Condition

- In the alternating optimization algorithm, we have

$$\mathbf{u}^{(k+1)} = (\mathbf{u}_1^{(k+1)}, \mathbf{u}_2^{(k+1)}) = (c_1(\mathbf{u}_2^{(k)}), c_2(c_1(\mathbf{u}_2^{(k)})))$$

for $k \geq 0$.

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We first prove that if f is concave, then the algorithm cannot be trapped at $\mathbf{u}$ if $f(\mathbf{u}) < f^*$.

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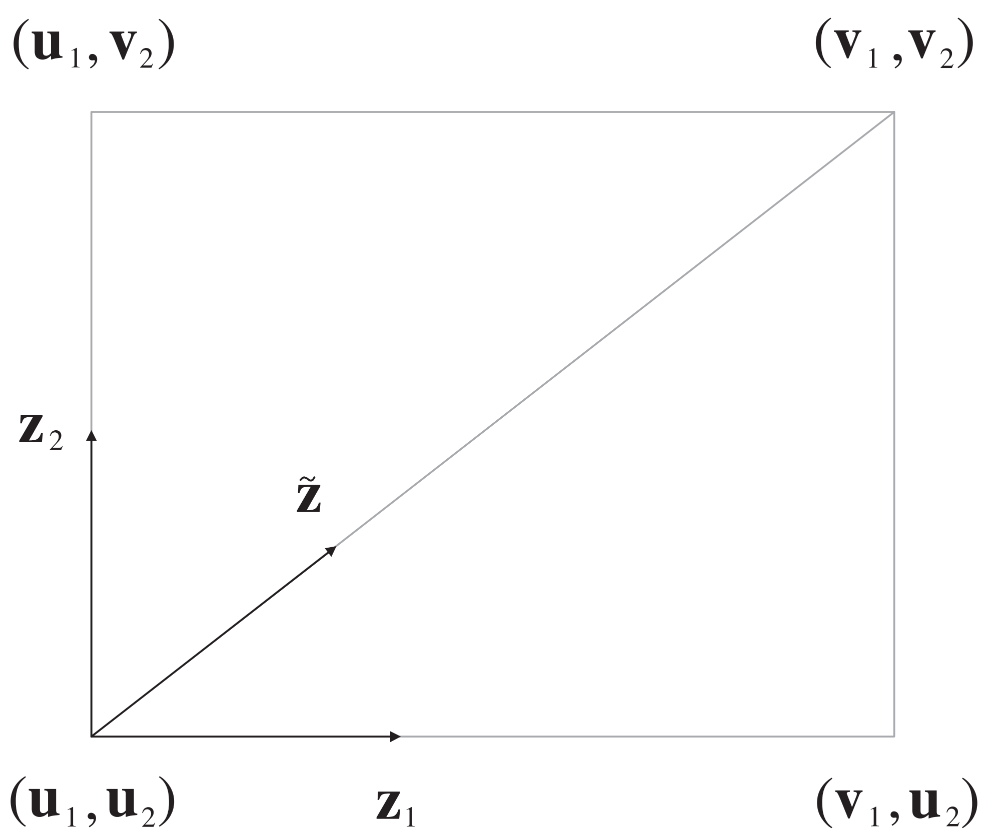

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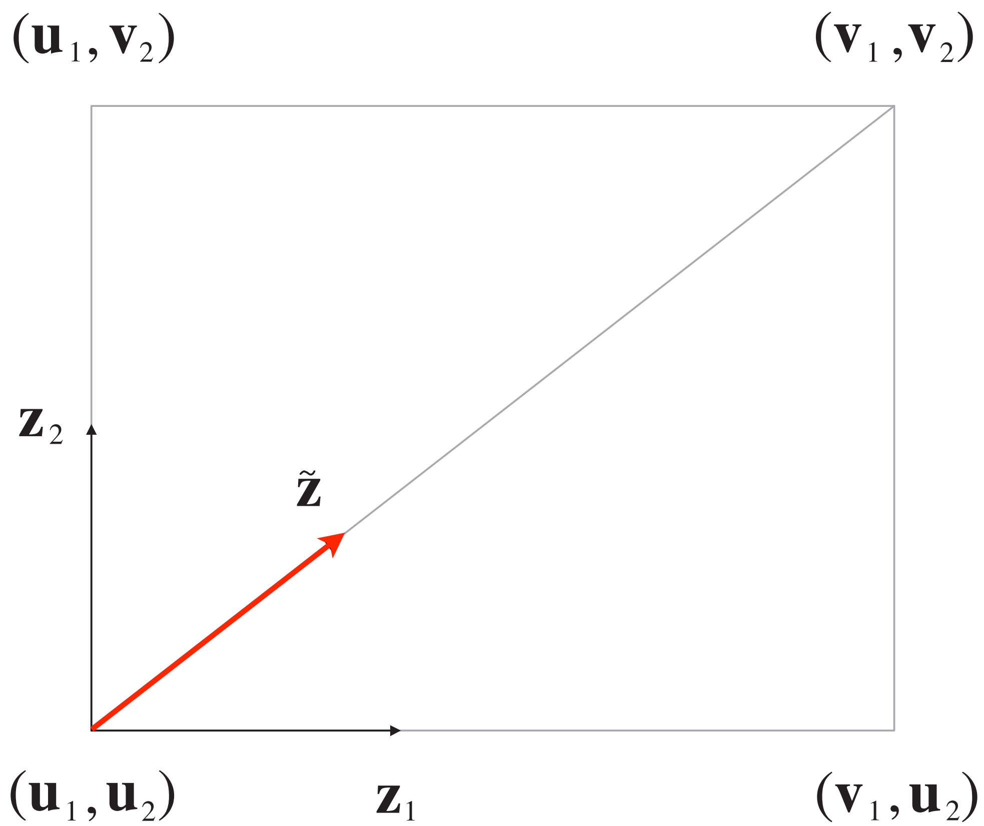

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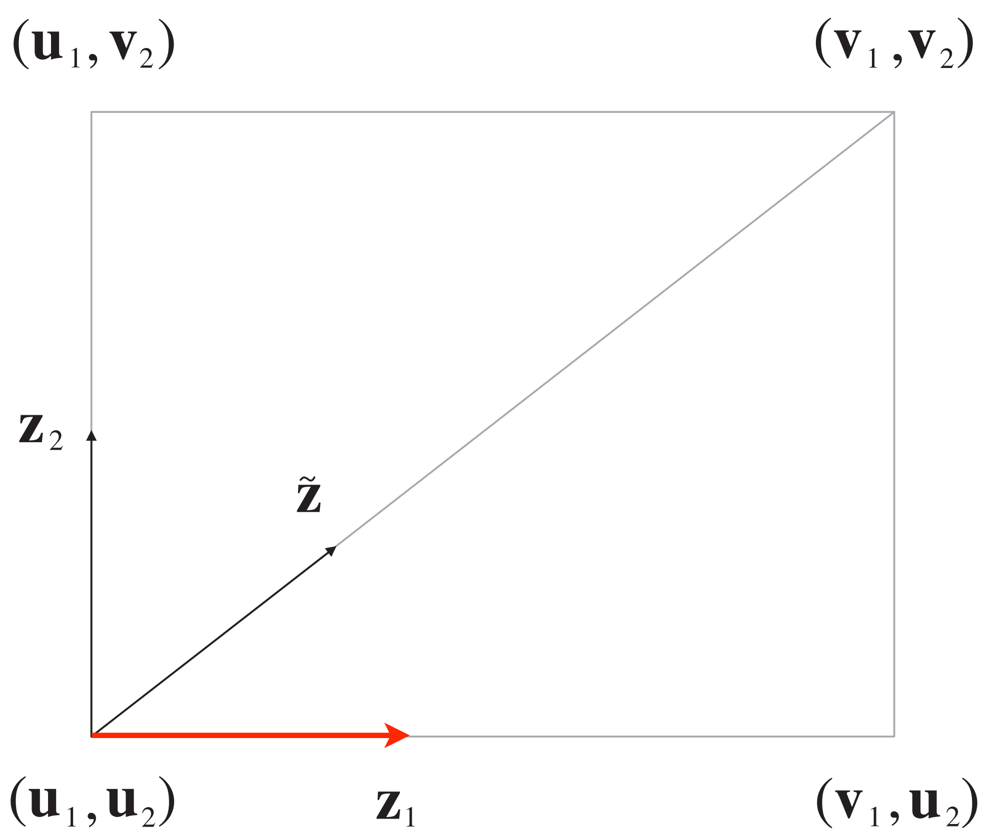

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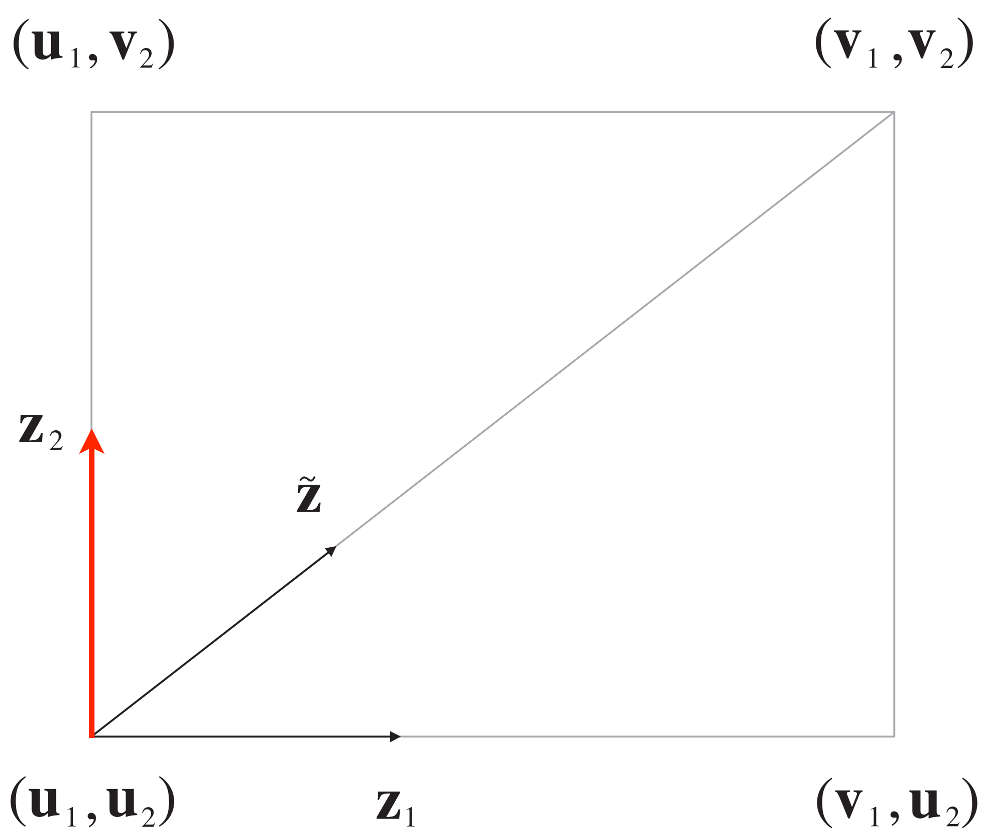

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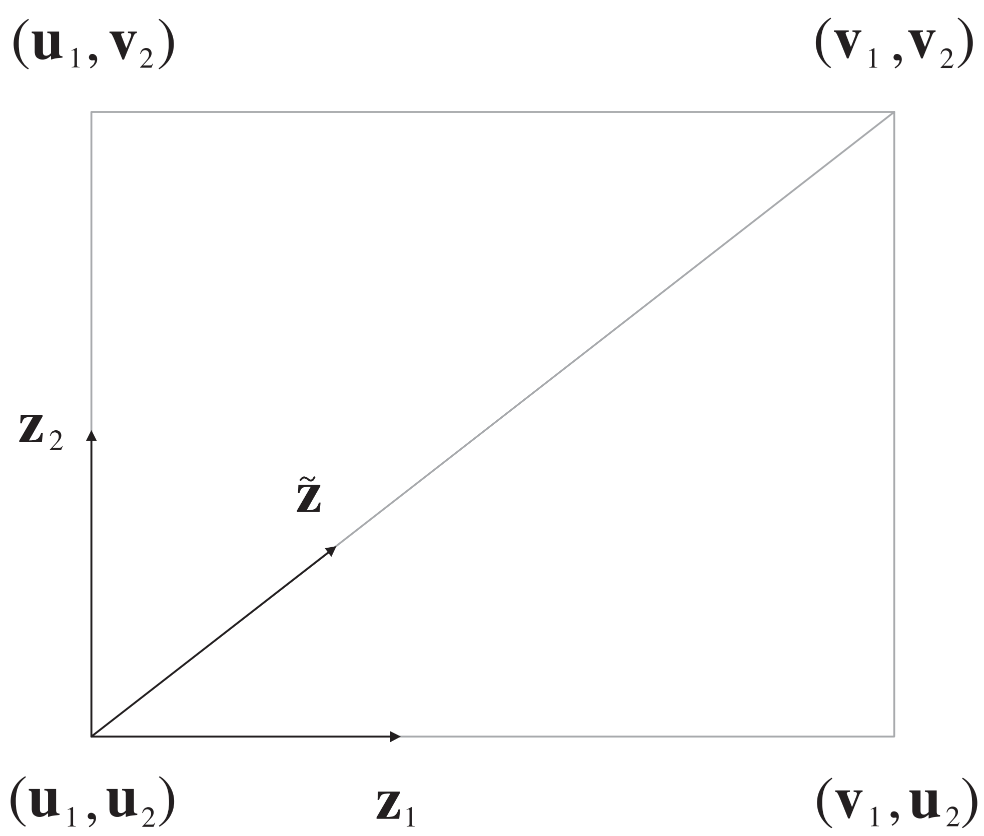

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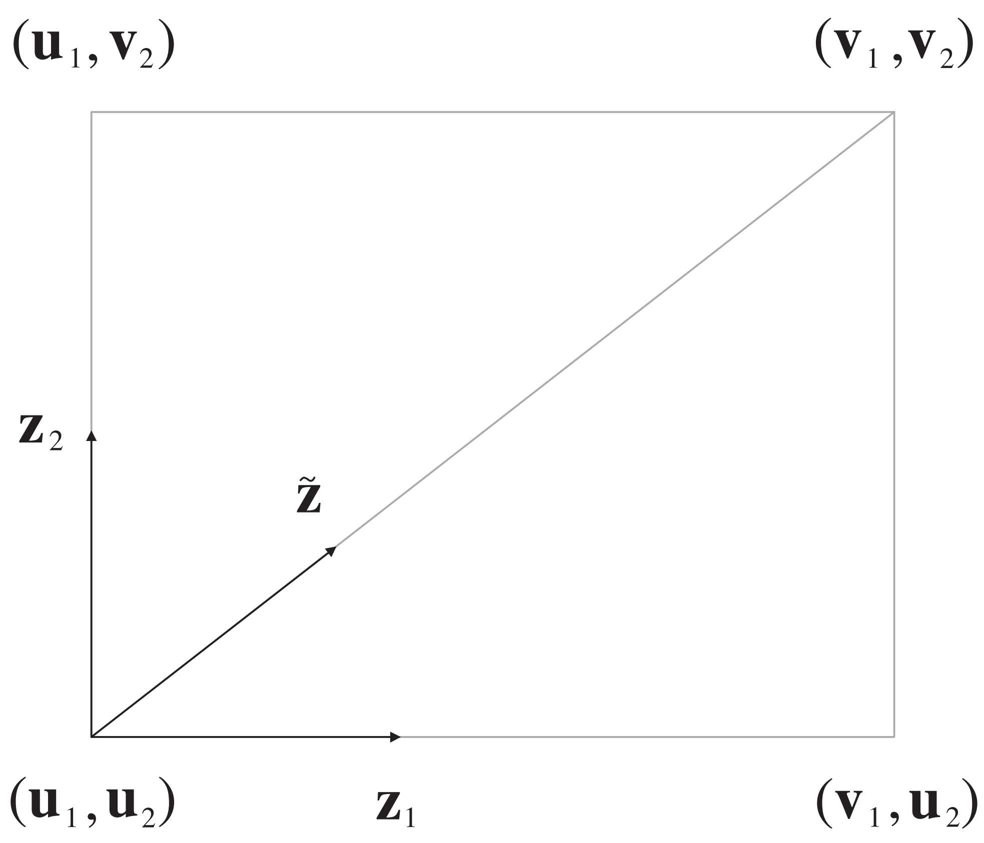

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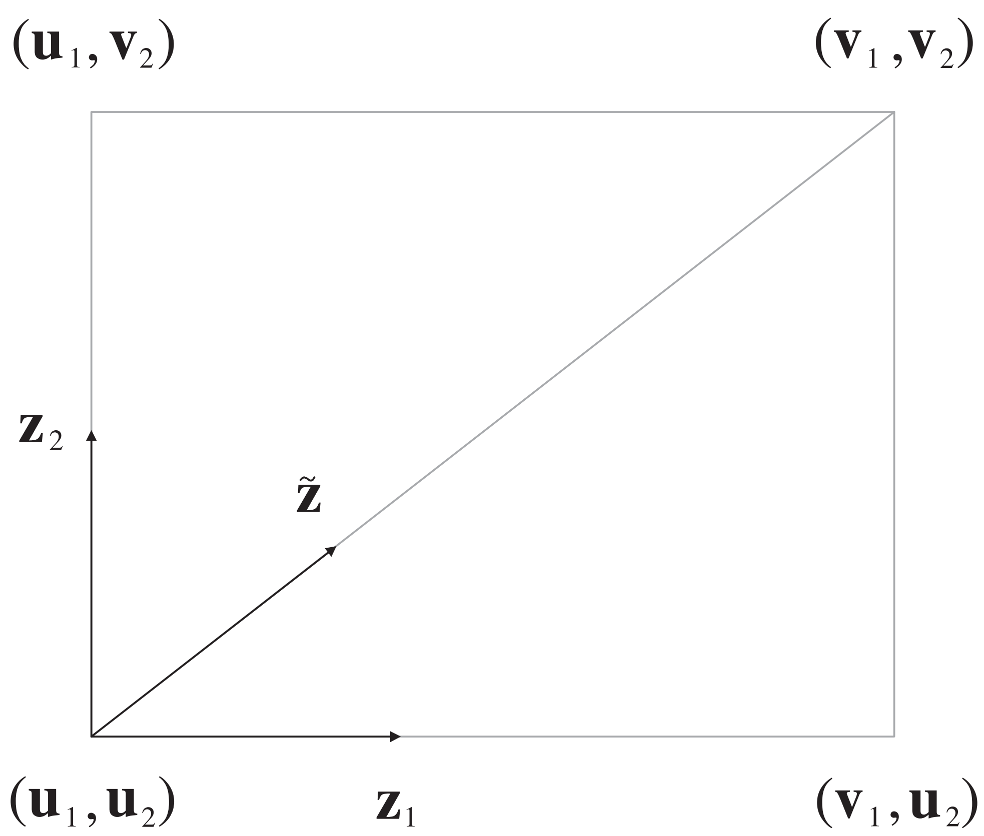

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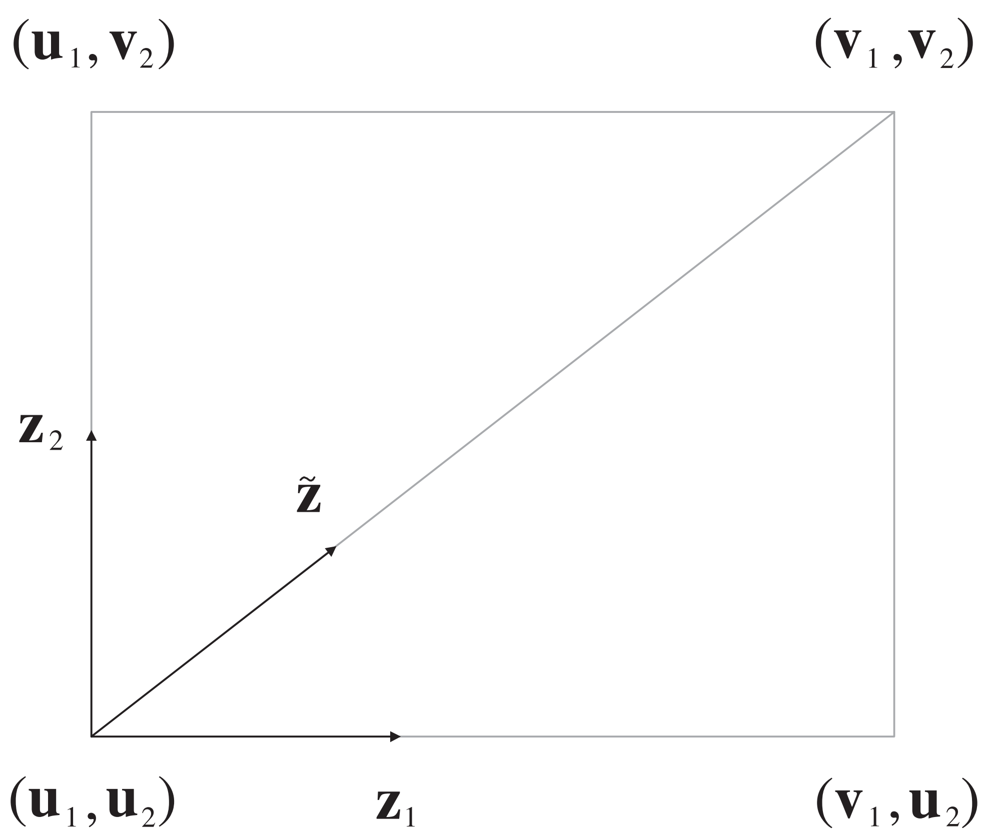

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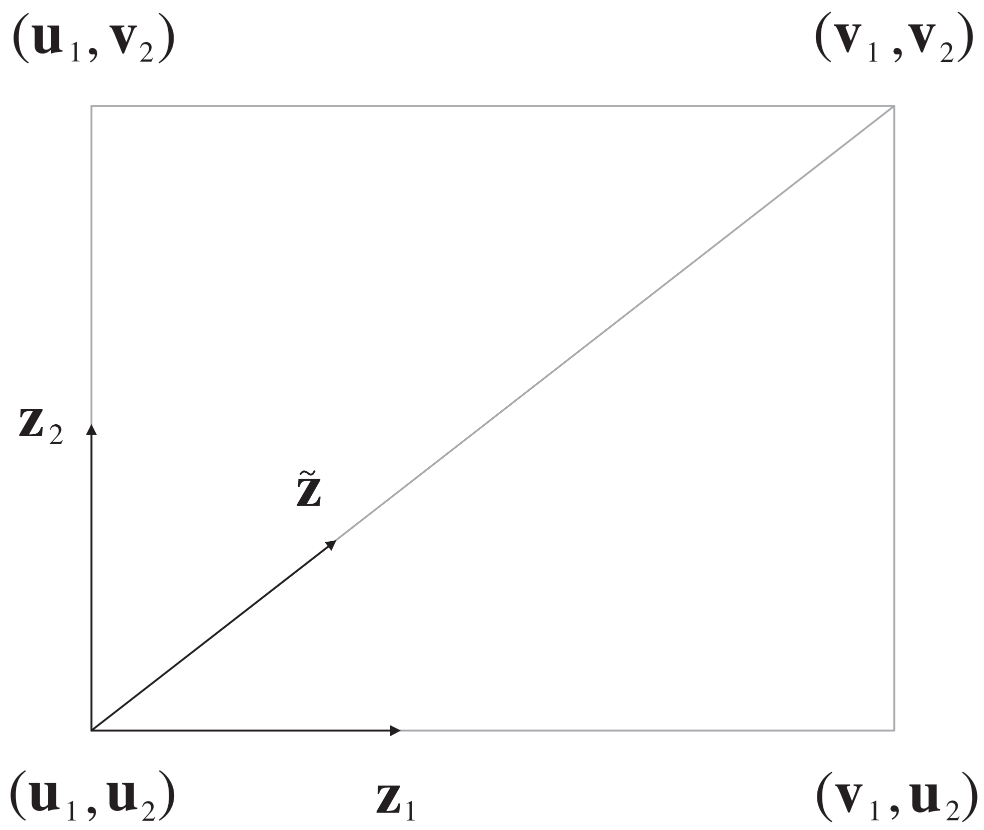

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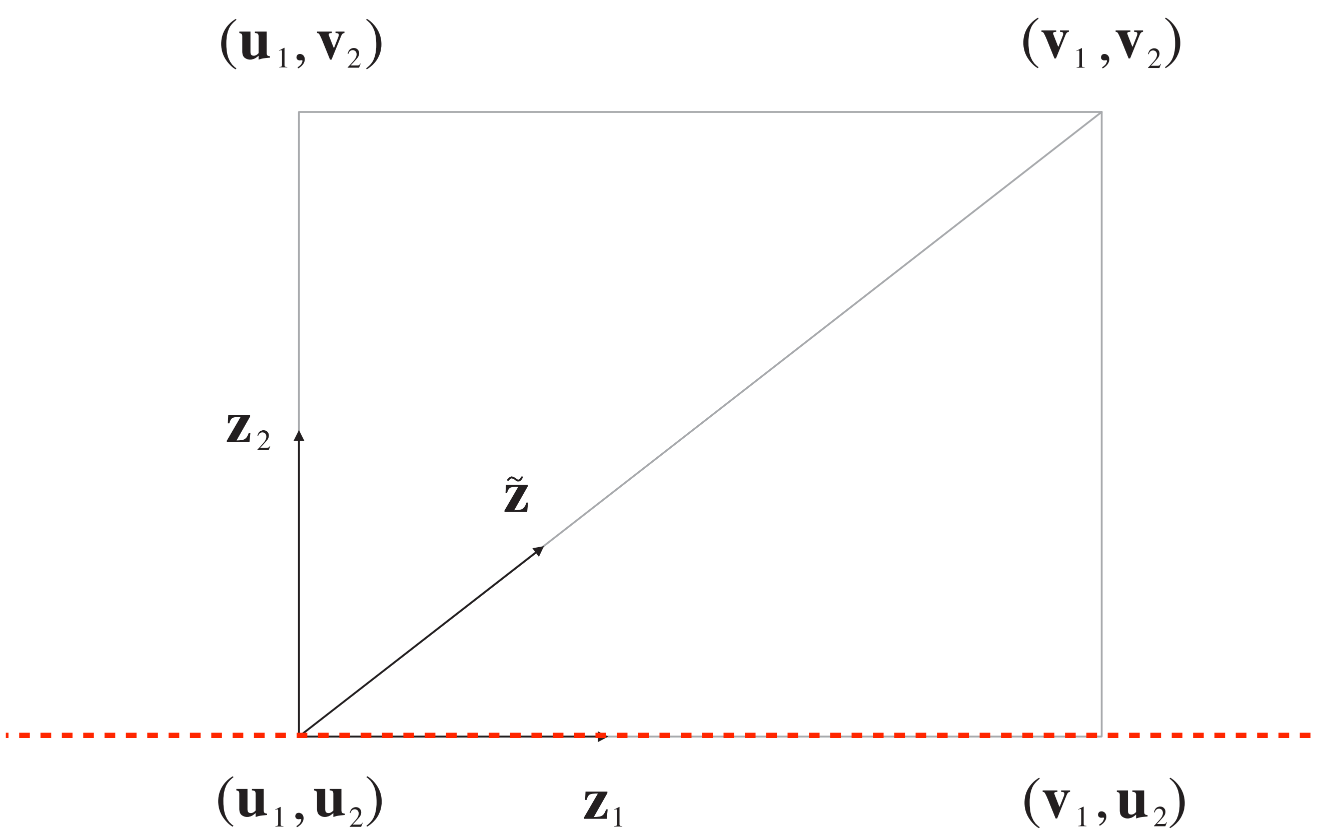

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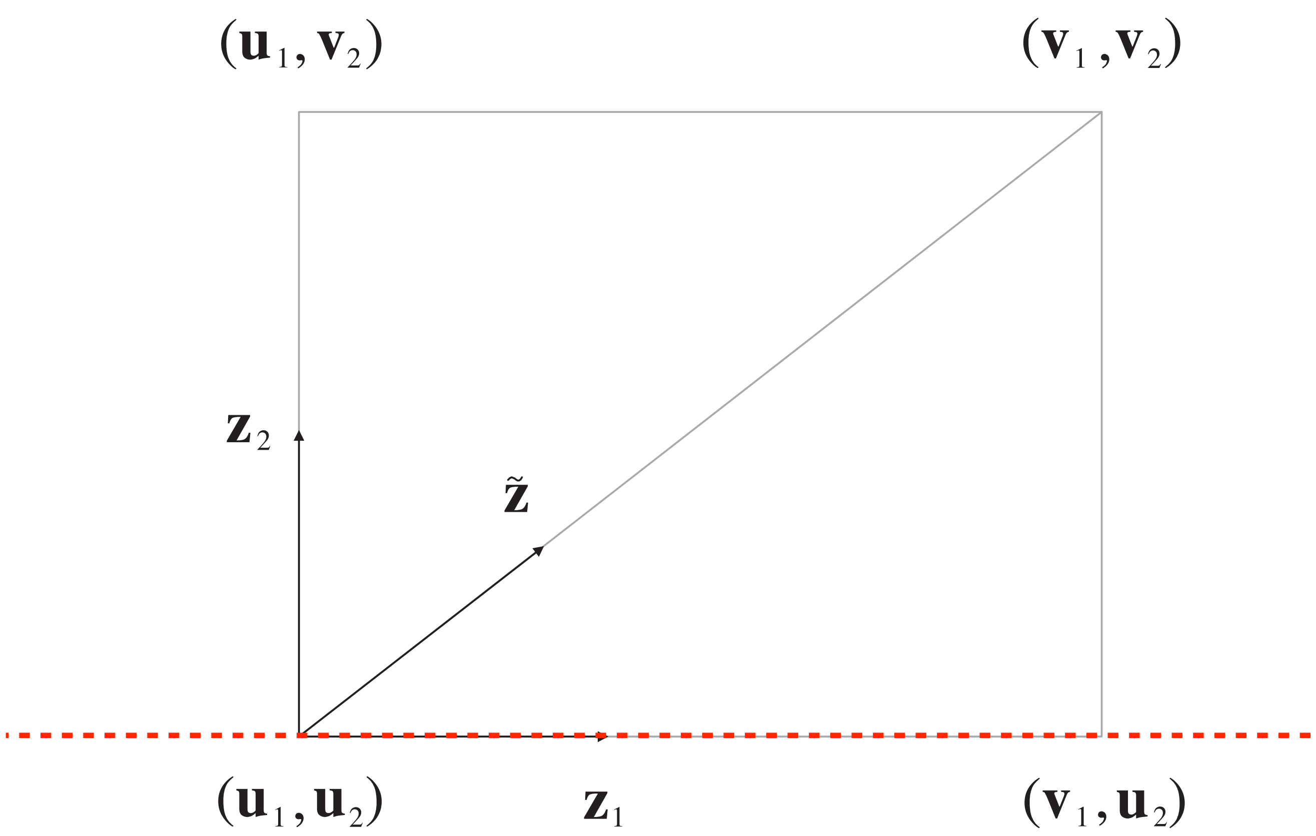

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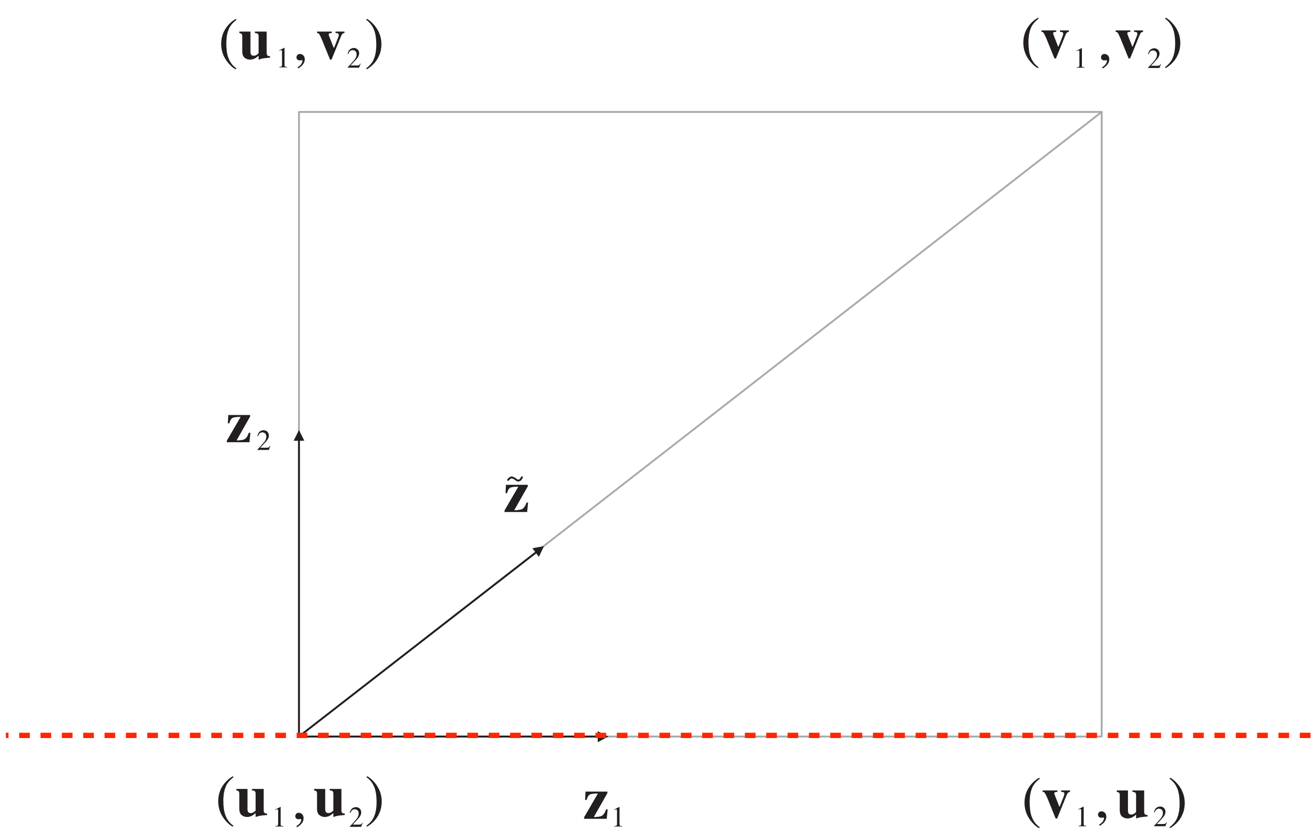

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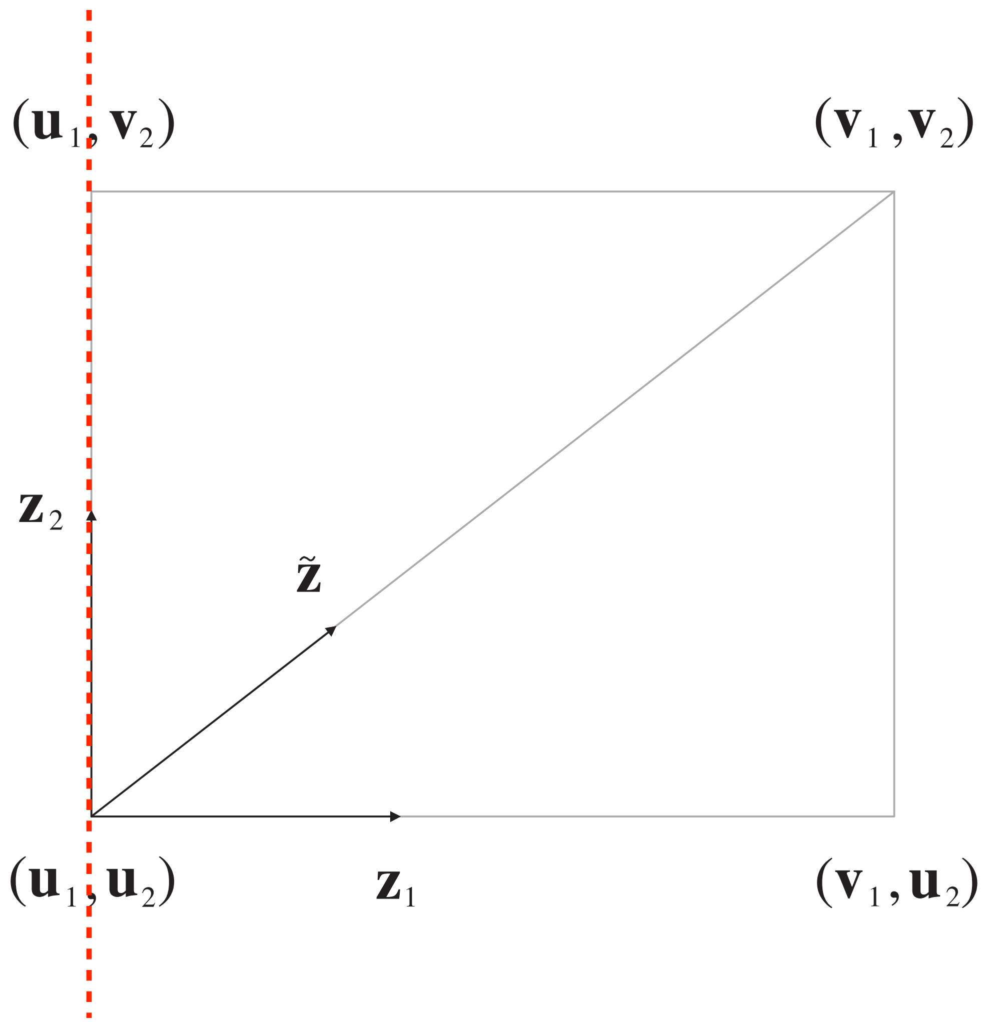

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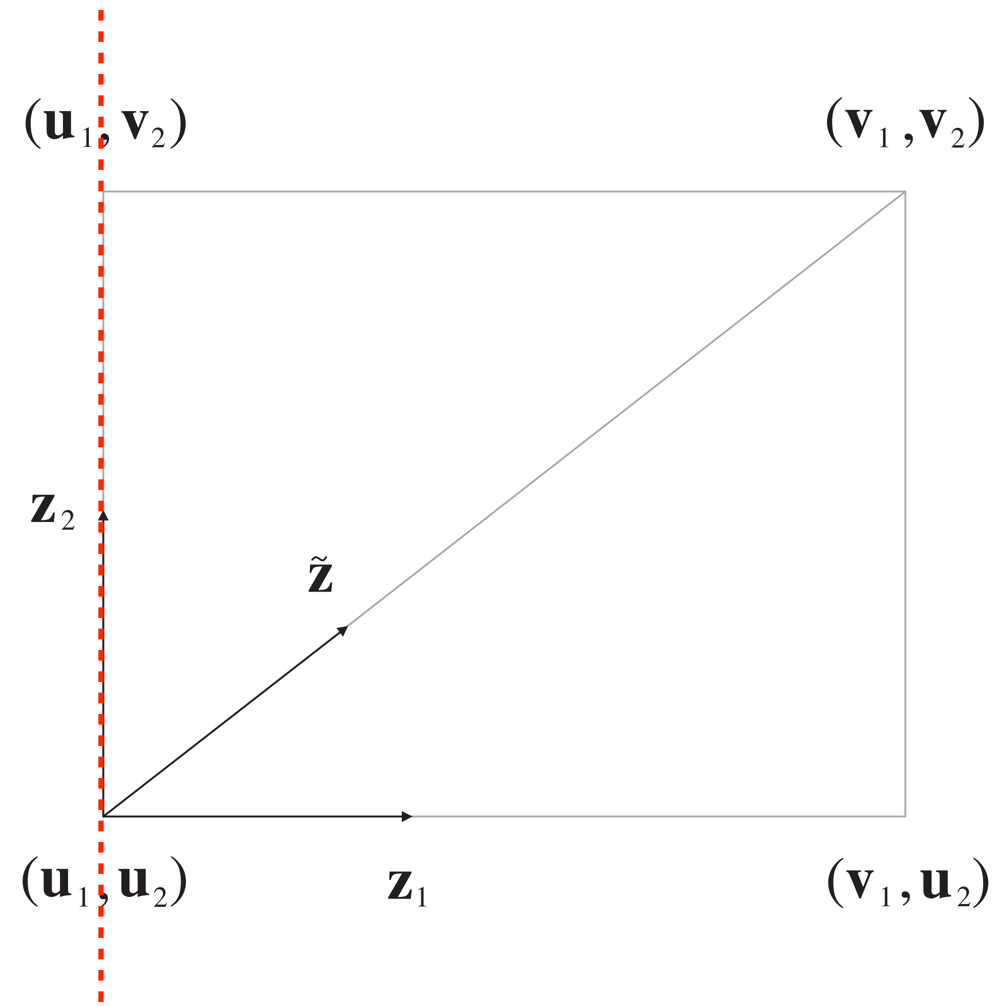

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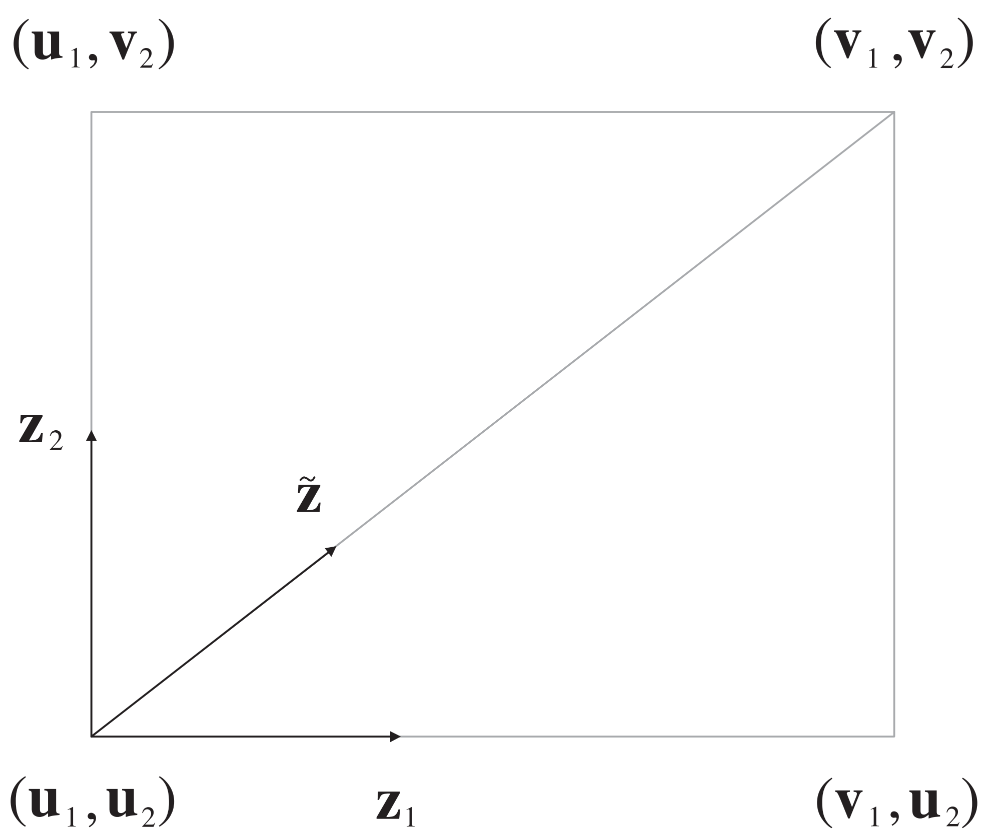

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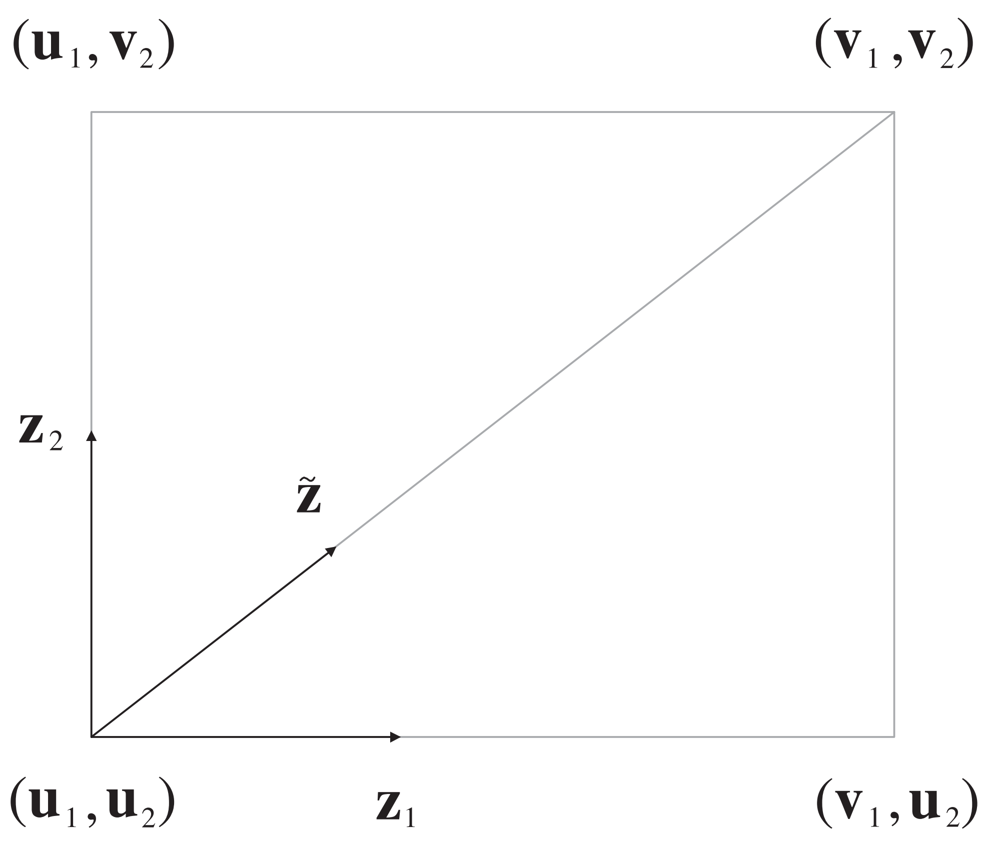

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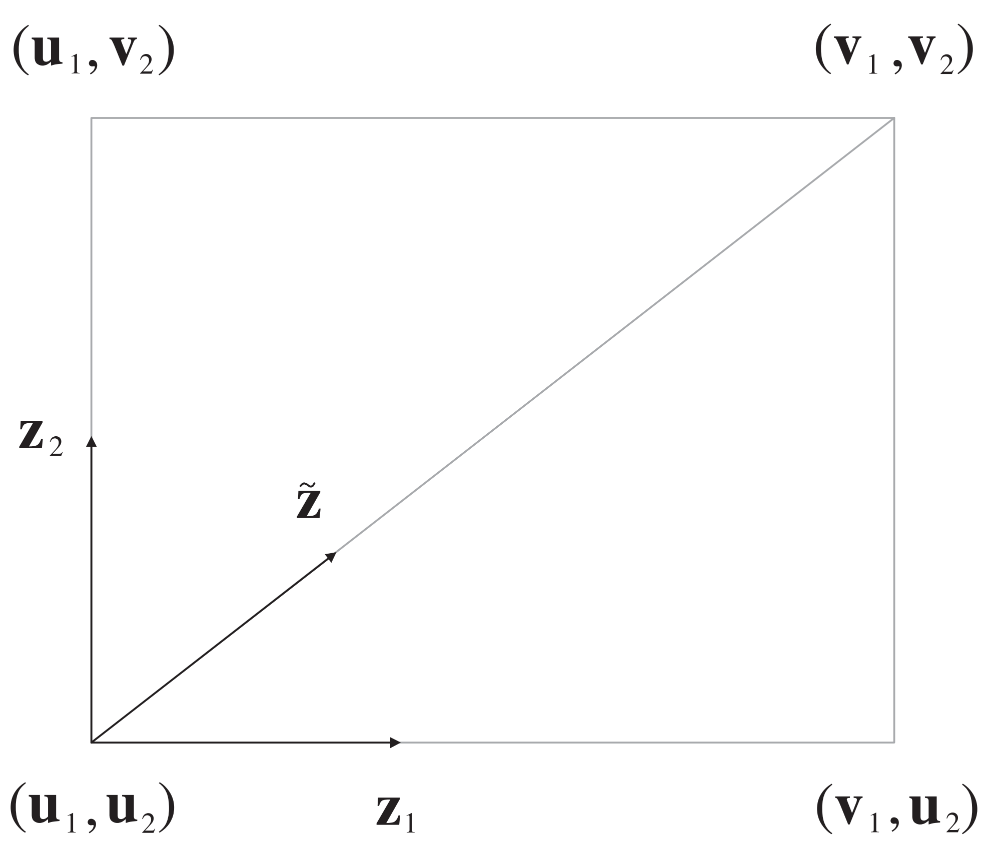

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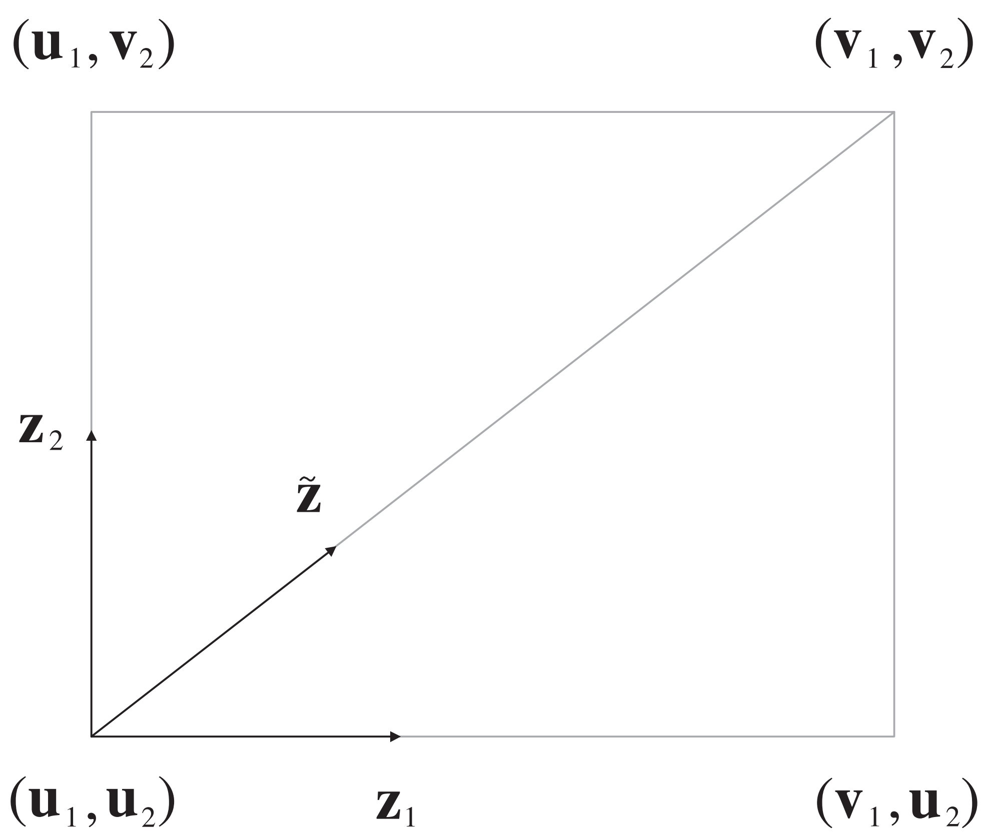

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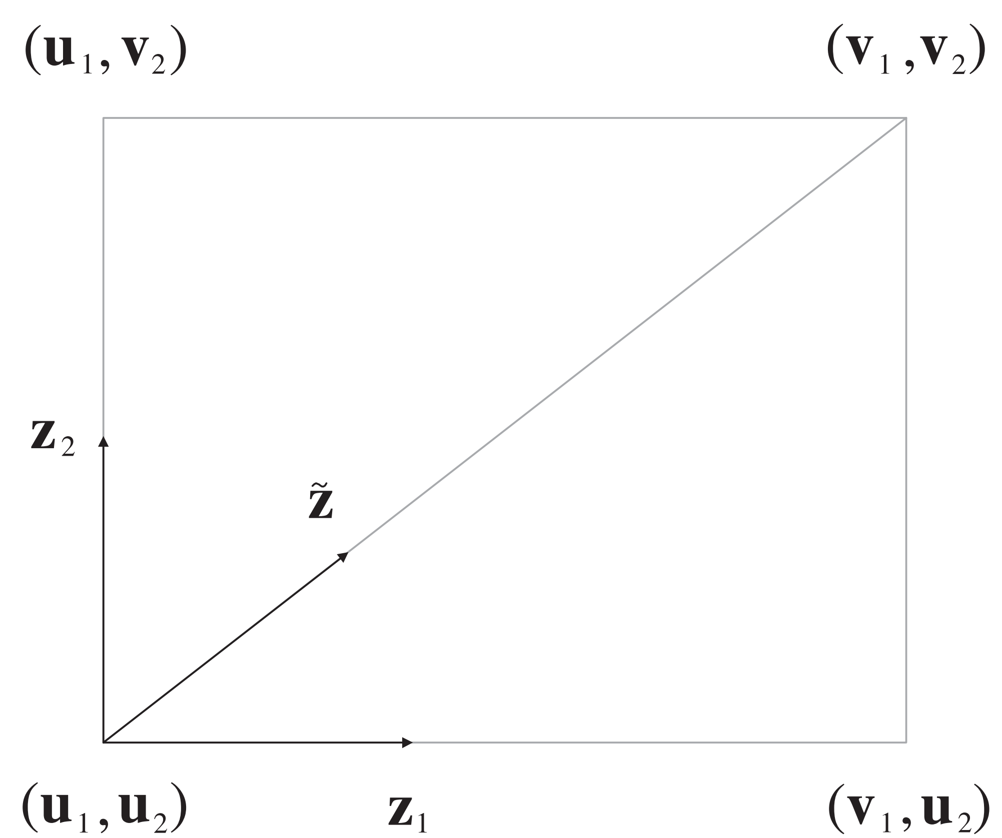

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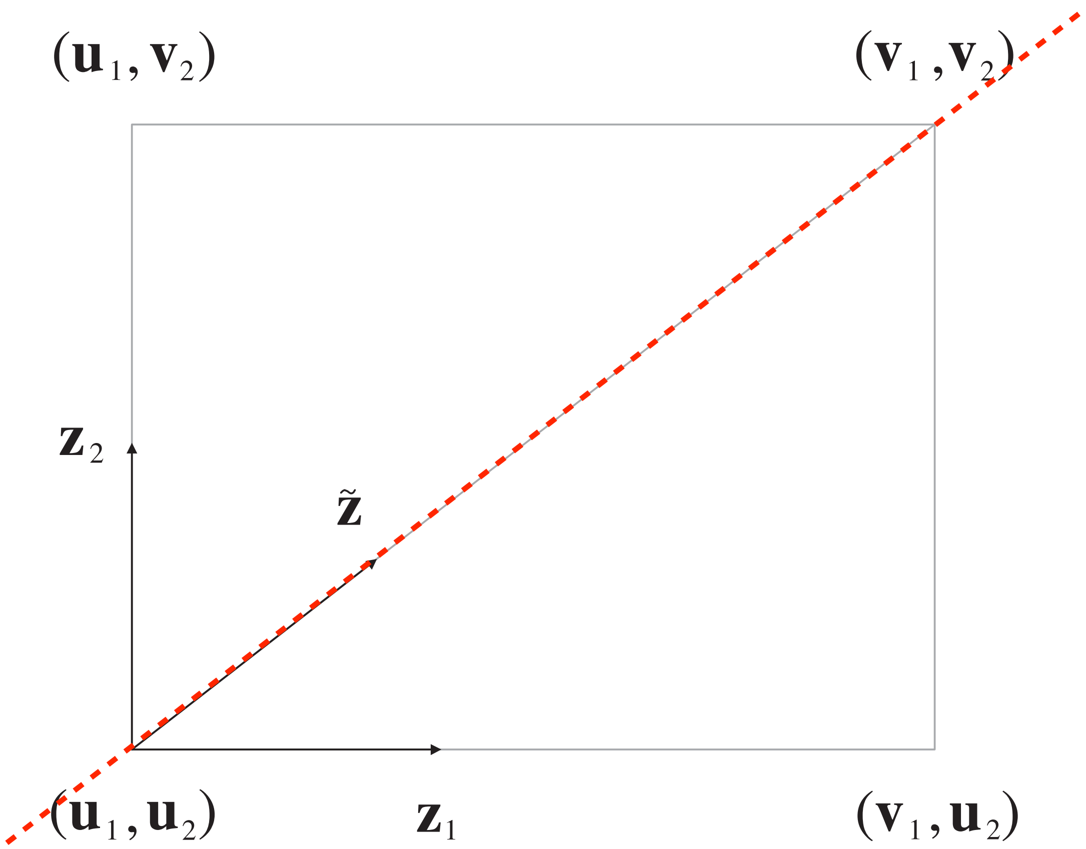

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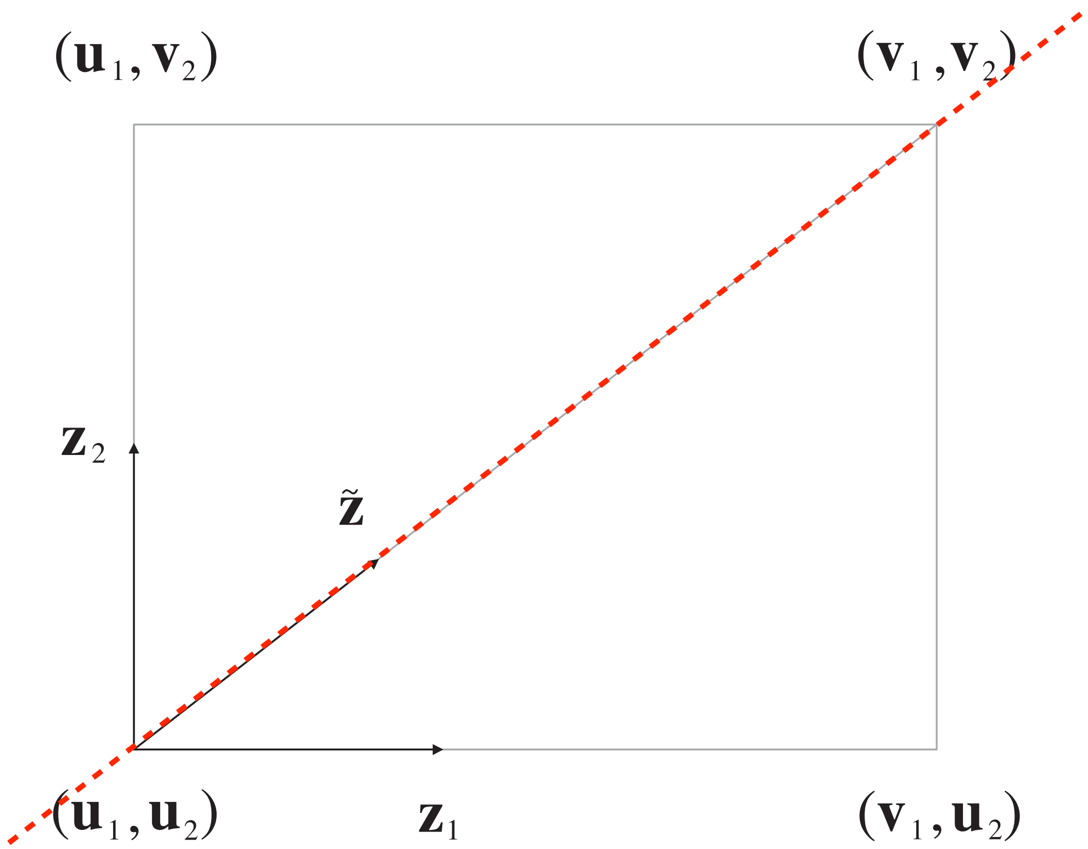

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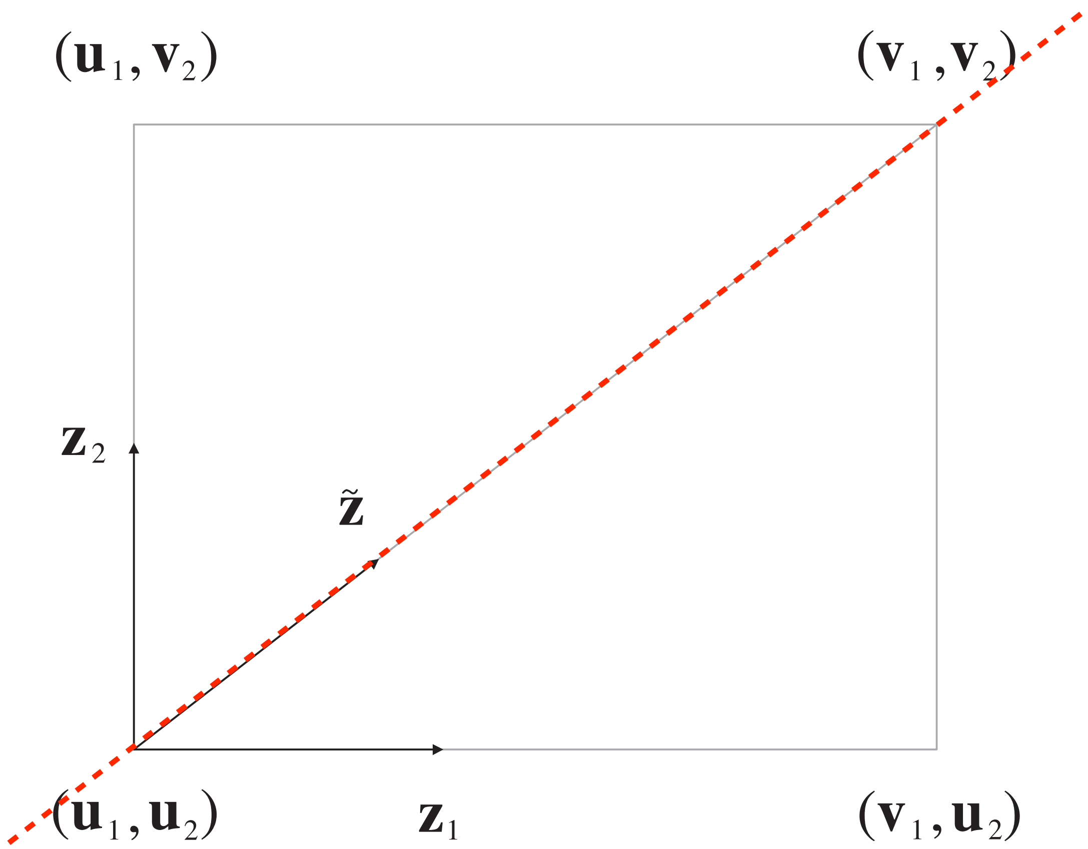

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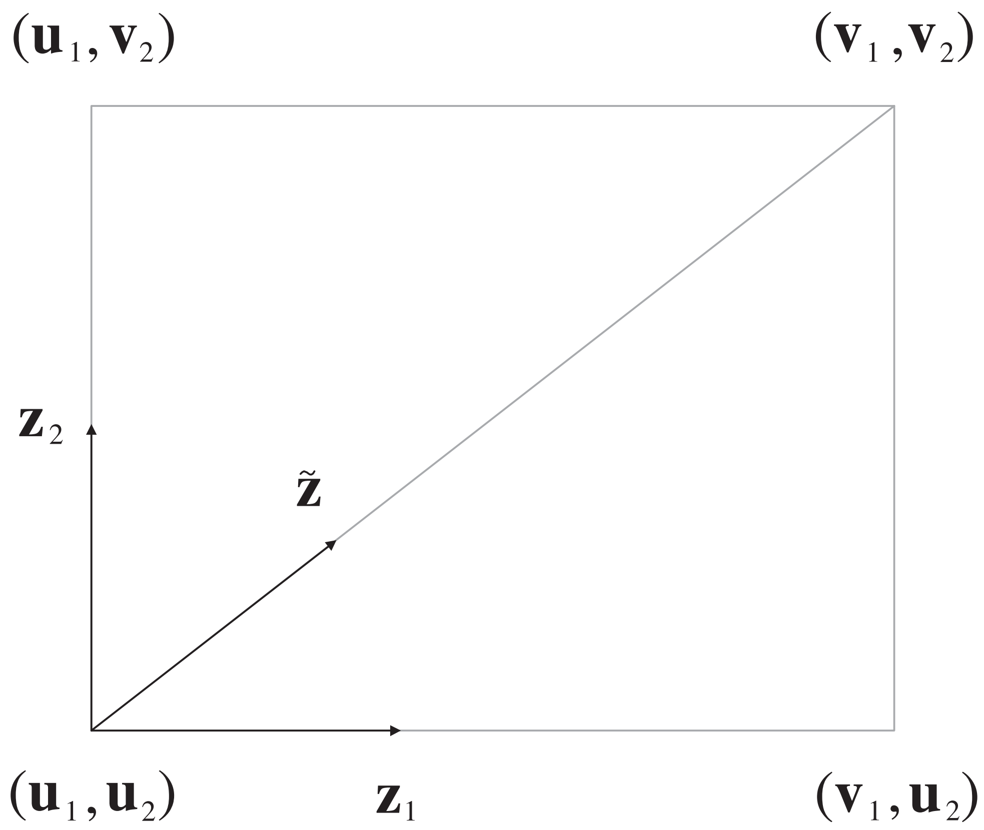

Although $\Delta f(\mathbf{u}) > 0$ as long as $f(\mathbf{u}) < f^*$, $f^{(k)}$ does not necessarily converge to f^* because the increment in $f^{(k)}$ in each step may be arbitrarily small.

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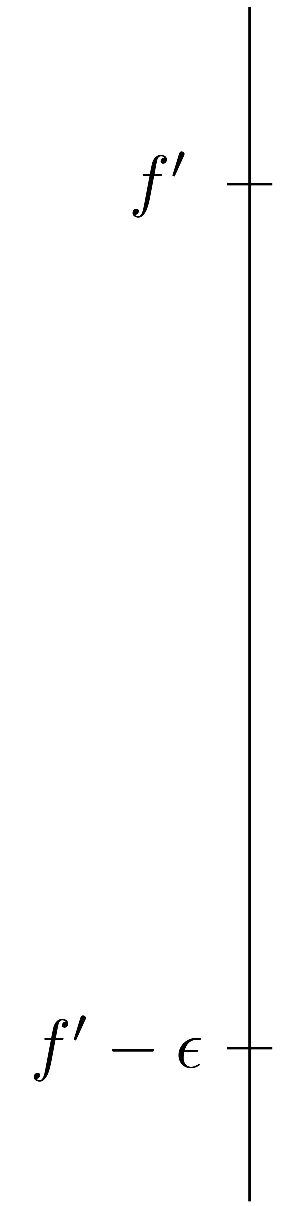

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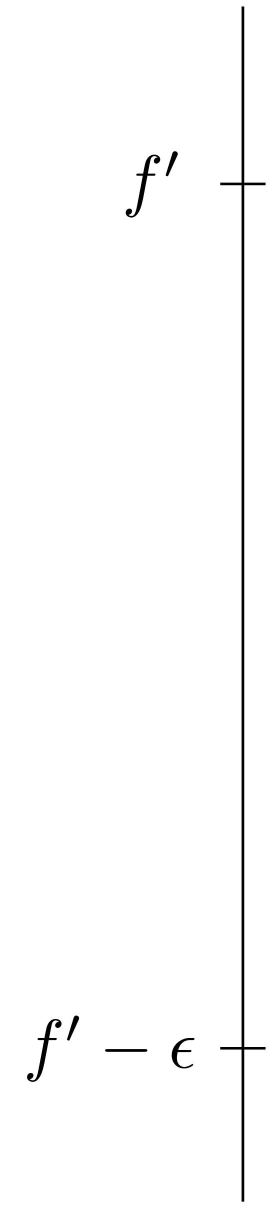

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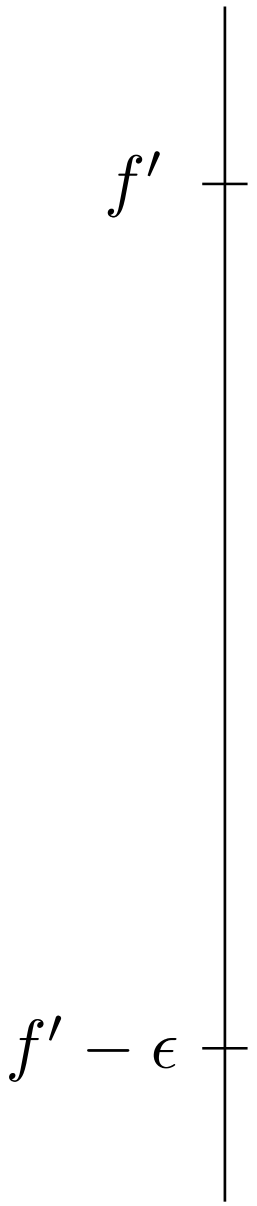

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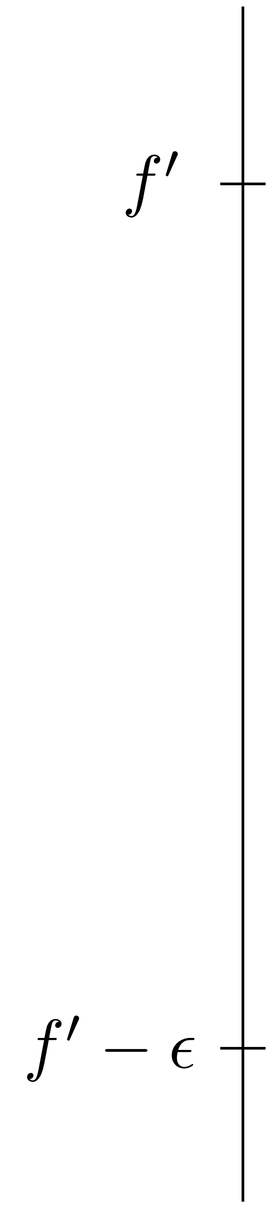

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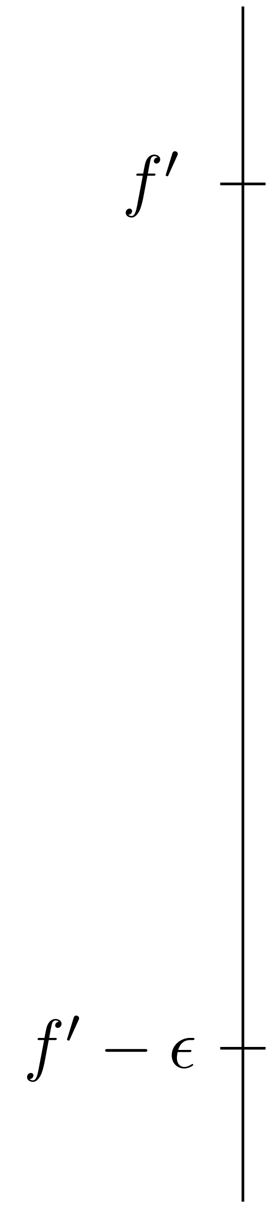

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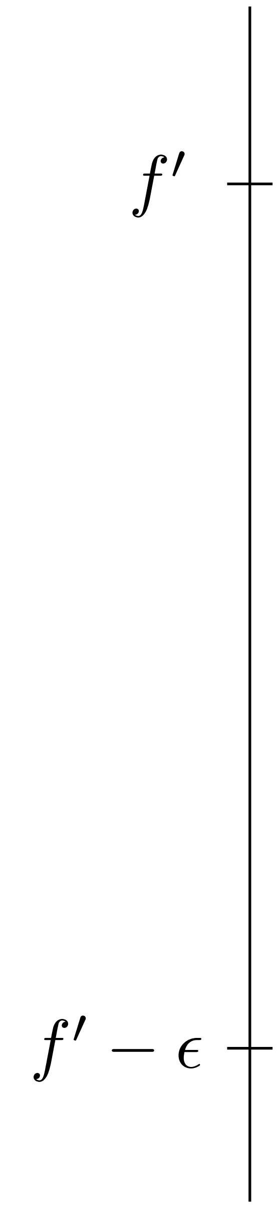

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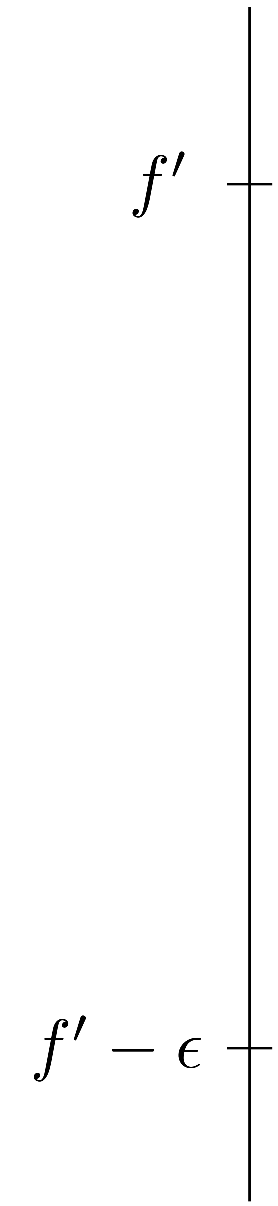

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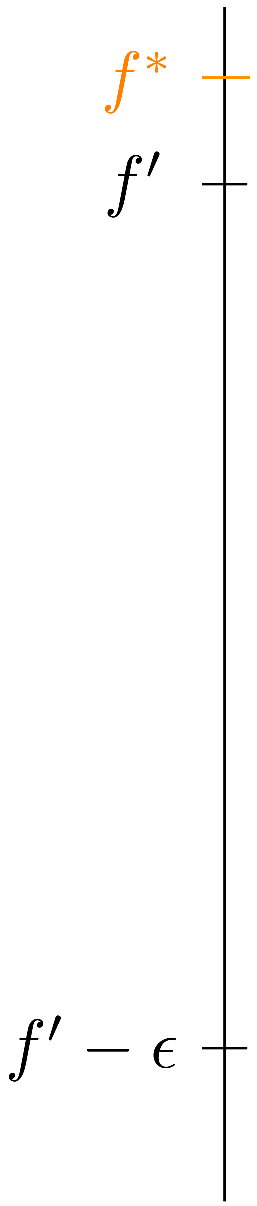

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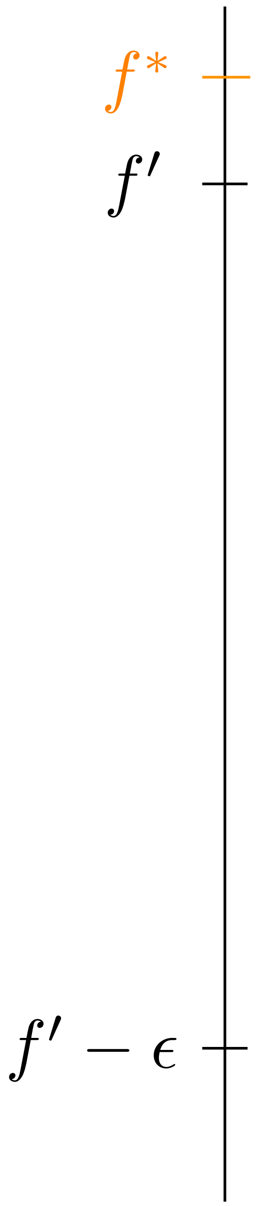

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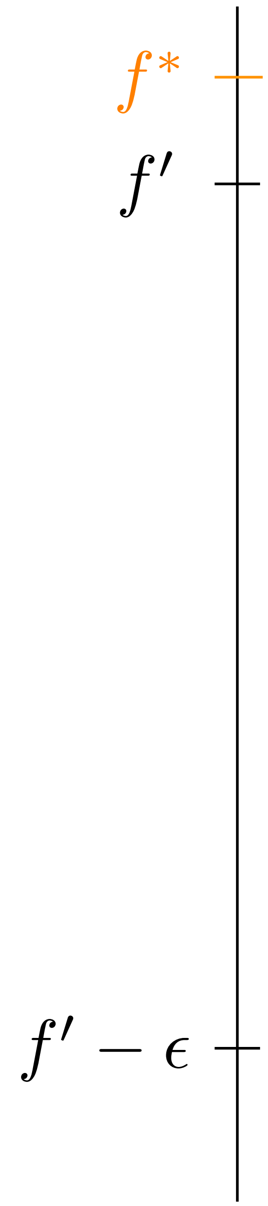

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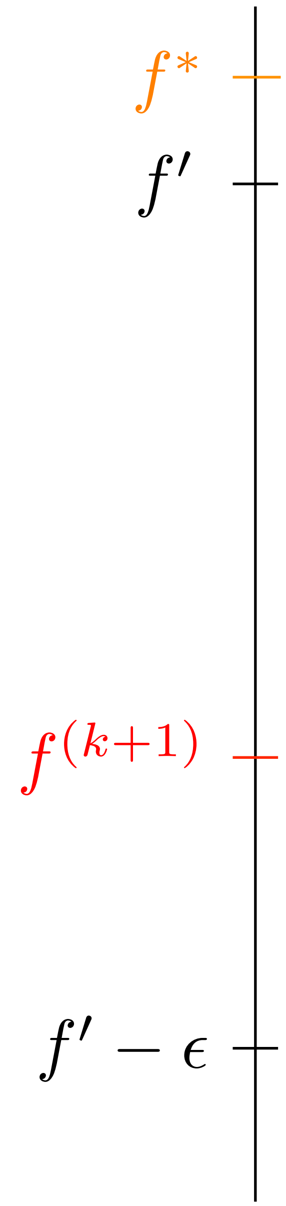

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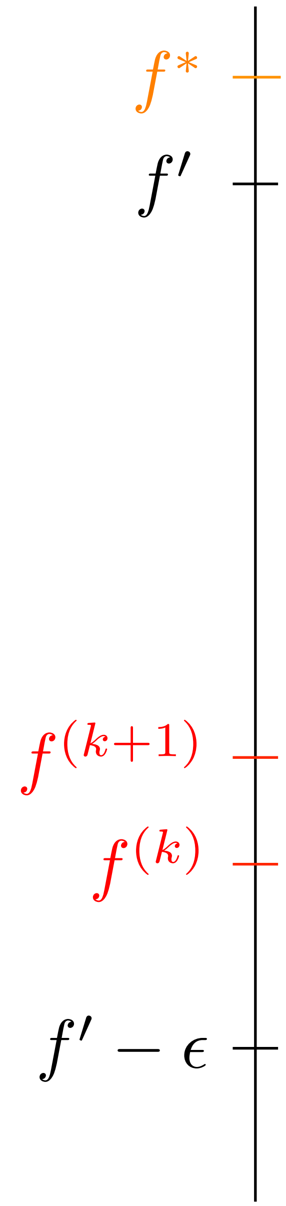

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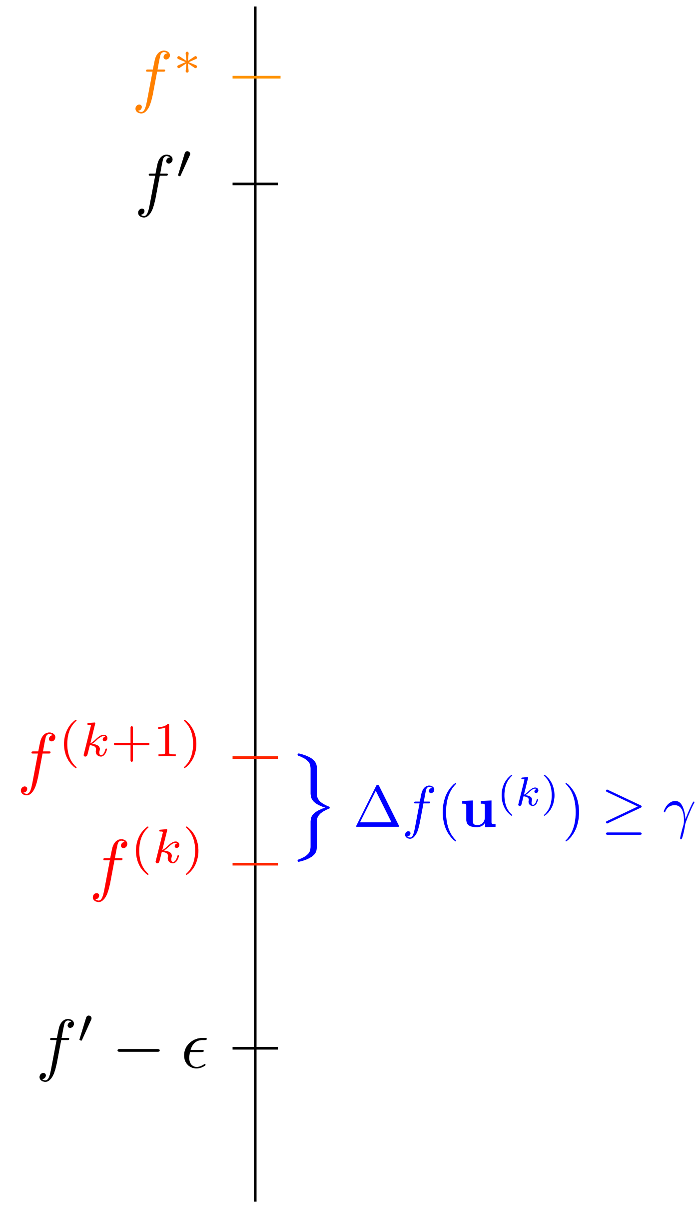

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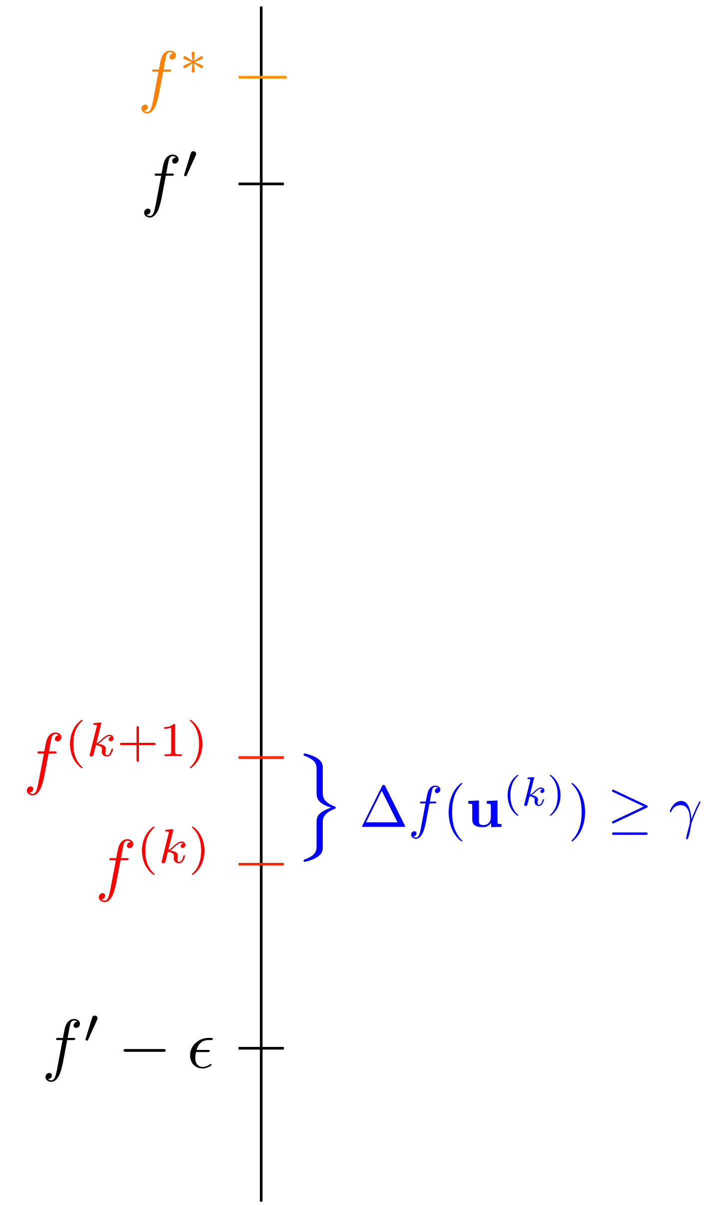

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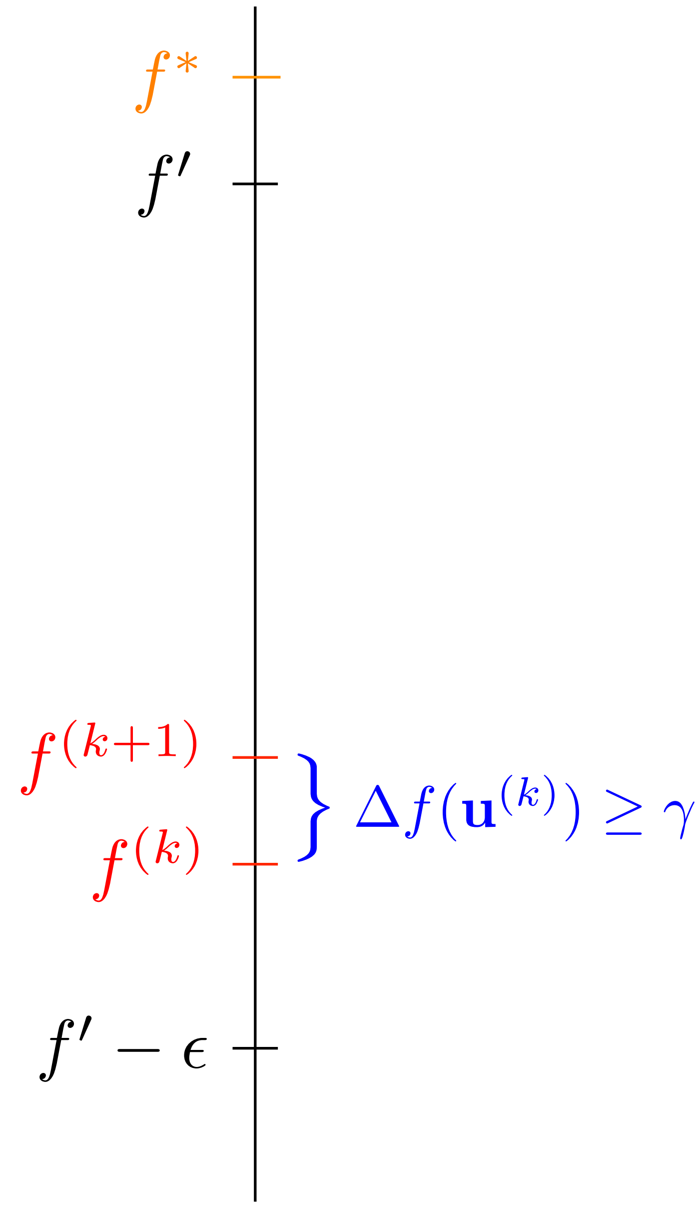

9.3.2 Convergence to the Channel Capacity

Concavity of *f*(*r*,*q*)

Proposition

$$f(\mathbf{r},\mathbf{q}) = \sum_x \sum_y r(x)p(y|x)\log\frac{q(x|y)}{r(x)}$$

is concave.

Proof

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$$(\lambda r_1(x) + \bar{\lambda} r_2(x)) \log \frac{\lambda r_1(x) + \bar{\lambda} r_2(x)}{\lambda q_1(x|y) + \bar{\lambda} q_2(x|y)} \le \lambda r_1(x) \log \frac{\cancel{\lambda} r_1(x)}{\cancel{\lambda} q_1(x|y)} + \bar{\lambda} r_2(x) \log \frac{\cancel{\bar{\lambda}} r_2(x)}{\cancel{\bar{\lambda}} q_2(x|y)}.$$

Concavity of *f*(*r*,*q*)

Proposition

$$f(\mathbf{r},\mathbf{q}) = \sum_x \sum_y r(x)p(y|x) \log \frac{q(x|y)}{r(x)}$$

is concave.

Proof

1. Consider $(\mathbf{r}_1, \mathbf{q}_1)$ and $(\mathbf{r}_2, \mathbf{q}_2)$ in A.
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3. Taking reciprocal in the logarithms yields

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